Sochastic Calculus via Stopping Derivatives

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Right derivatives

- The right derivative of $f : [0, \infty) \to \mathbb{R}$ at $s \in [0, \infty)$ is

$$\lim_{t \downarrow s} \frac{f(t) - f(s)}{t - s} \qquad \text{(limit as } t \text{ tends to } s \text{ from above)}$$

- If $f : [0, \infty) \to \mathbb{R}$ has right derivative d at s and $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $f(s)$ then $g \circ f$ has right derivative $g'(f(s)) \cdot d$ at s.

- Let $f : [0, \infty) \to \mathbb{R}$ be measurable such that $\int_0^t f(s)\, ds < \infty$ for all t. If f is right-continuous at s, then the continuous function $t \mapsto \int_0^t f(s)\, ds$ has right derivative $f(s)$ at s.

Stochastic integral

Let $(W_t)_t$ be a Brownian motion and consider

$$X_t \;:=\; C_0 + \int_0^t B_s\,ds + \int_0^t \sigma_s\,dW_s$$

- B_s is the drift of $(X_t)_t$ at time s.
- σ_s is the diffusion coefficient of $(X_t)_t$ at time s.

Can we recover B_s and $A_s := {\sigma_s}^2$ as appropriate forms of derivative of $(X_t)_t$ at s?

([Allouba 2007] recovers σ_s using a form of derivative involving the quadratic covariation of X_t and W_t.)

- The drift B_s should be some sort of right derivative at s for conditional expectation

$$t \mapsto \mathrm{E}[X_t \mid \mathcal{F}_s] \qquad \text{for times } t \geq s$$

- The variance rate A_s should be some sort of right derivative at s for conditional variance

$$t \mapsto \mathrm{Var}[X_t \mid \mathcal{F}_s] \qquad \text{for times } t \geq s$$

where $\mathrm{Var}[X_t \mid \mathcal{F}_s] := \mathrm{E}[(X_t - \mathrm{E}[X_t \mid \mathcal{F}_s])^2 \mid \mathcal{F}_s]$

(Similar ideas are followed in, e.g., [Kolmogorov 1933, Doeblin 1940, Nelson 1967].)

- The drift B_s at time s is a right derivative at s for conditional expectation

$$T \mapsto \mathrm{E}[X_T \mid \mathcal{F}_s] \qquad \text{for finite stopping times } T \geq s$$

- The variance rate A_s at time s is a right derivative at s for conditional variance

$$T \mapsto \mathrm{Var}[X_T \mid \mathcal{F}_s] \qquad \text{for finite stopping times } T \geq s$$

where $\mathrm{Var}[X_T \mid \mathcal{F}_s] := \mathrm{E}[(X_T - \mathrm{E}[X_T \mid \mathcal{F}_s])^2 \mid \mathcal{F}_s]$

- The drift B_S at a finite stopping time S is a right derivative at S for conditional expectation

$$T \mapsto \mathrm{E}[X_T \mid \mathcal{F}_S] \qquad \text{for finite stopping times } T \geq S$$

- The variance rate A_S at a finite stopping time S is a right derivative at S for conditional variance

$$T \mapsto \mathrm{Var}[X_T \mid \mathcal{F}_S] \qquad \text{for finite stopping times } T \geq S$$

where $\mathrm{Var}[X_T \mid \mathcal{F}_S] := \mathrm{E}[(X_T - \mathrm{E}[X_T \mid \mathcal{F}_S])^2 \mid \mathcal{F}_S]$

$\mathrm{E}[X_T \mid \mathcal{F}_S]$ and $\mathrm{Var}[X_T \mid \mathcal{F}_S]$ are examples of stopping functionals

$$S \leq T \ \mapsto \ F[T \mid \mathcal{F}_S]$$

- For finite stopping times $S \leq T$, it holds that $F[T \mid \mathcal{F}_S]$ (if defined) is an $\mathcal{F}_S$-measurable random variable .
- Suppose $(T_i)_{i \in I}$ is a countable family of finite stopping times $> S$ such that $F[T_i \mid \mathcal{F}_S]$ is defined for all $i \in I$ and suppose that $\{Q_i \mid i \in I\}$ is an $\mathcal{F}_S$-measurable partition of Ω. Then

$$F\left[\sum_{i \in I} \mathbb{1}_{Q_i} \cdot T_i \,\middle|\, \mathcal{F}_S\right] = \sum_{i \in I} \mathbb{1}_{Q_i} \cdot F[T_i \mid \mathcal{F}_S] \ .$$

Stopping derivative

An $\mathcal{F}_S$-measurable random variable D_S is a stopping derivative for a stopping functional F at a finite stopping time S if

$$D_S = \lim_{T \downarrow S} \frac{F[T \mid \mathcal{F}_S] - F[S \mid \mathcal{F}_S]}{\mathrm{E}[T - S \mid \mathcal{F}_S]}$$

Here the limit is a limit over a net of random variables given by the stopping functional

$$G[T \mid \mathcal{F}_S] := \frac{F[T \mid \mathcal{F}_S] - F[S \mid \mathcal{F}_S]}{\mathrm{E}[T - S \mid \mathcal{F}_S]}$$

and indexed by finite stopping times T with $T > S$.

There is only one notion of limit

The theorem below defines

$$D_S \;:=\; \lim_{T \downarrow S} G[T \mid \mathcal{F}_S]$$

Theorem The following are equivalent, for any stopping functional G, finite stopping time S and $\mathcal{F}_S$-measurable random variable D_S

1. (Strong uniform convergence) For any $\mathcal{F}_S$-measurable random variable $\underline{\varepsilon} > 0$, there exists $U > S$ such that, for any finite T with $S < T \leq U$, it holds that $|G[T \mid \mathcal{F}_S] - D_S| \leq \underline{\varepsilon}$ a.s.
2. (Uniform convergence) For any real $\varepsilon > 0$, there exists $U > S$ such that, for any finite T with $S < T \leq U$, it holds that $|G[T \mid \mathcal{F}_S] - D_S| \leq \varepsilon$ a.s.
3. (Convergence in probability) For any real $\varepsilon > 0$, it holds that $\lim_{T \downarrow S} \mathrm{P}[|G[T \mid \mathcal{F}_S] - D_S| \leq \varepsilon] = 1$ (net convergence).

- The drift B_S at a finite stopping time S is a right derivative at S for conditional expectation

$$T \mapsto \mathsf{E}[X_T \mid \mathcal{F}_S] \qquad \text{for finite stopping times } T \geq S$$

B_S is a stopping derivative for $\mathsf{E}[X_T \mid \mathcal{F}_S]$ at S.

- The variance rate A_S at a finite stopping time S is a right derivative at S for conditional variance

$$T \mapsto \mathrm{Var}[X_T \mid \mathcal{F}_S] \qquad \text{for finite stopping times } T \geq S$$

where $\mathrm{Var}[X_T \mid \mathcal{F}_S] := \mathsf{E}[(X_T - \mathsf{E}[X_T \mid \mathcal{F}_S])^2 \mid \mathcal{F}_S]$

A_S is a stopping derivative for $\mathrm{Var}[X_T \mid \mathcal{F}_S]$ at S.

Robustness of variance rate

Theorem Suppose $(X_t)_t$ has drift at S. Then the variance rate at S can be equivalently defined as:

- a stopping derivative for conditional variance $\mathrm{Var}[X_T \mid \mathcal{F}_S]$,
- a stopping derivative for relative second moment $\mathrm{E}[(X_T - X_S)^2 \mid \mathcal{F}_S]$.

A fundamental theorem of calculus for stopping derivatives

Theorem Suppose

$$X_t \;=\; C_0 + \int_0^t B_s\, ds + \int_0^t \sigma_s\, dW_s$$

where $(B_t)_t$ and $(\sigma_t)_t$ are right-continuous adapted processes such that $\int_0^t B_s(\omega)\, ds < \infty$ and $\int_0^t (\sigma_s(\omega))^2\, ds < \infty$, for all $t > 0$ and $\omega \in \Omega$. Let S be a finite stopping time. Then

- B_S is a stopping derivative for $\mathrm{E}[X_T \mid \mathcal{F}_S]$ at S, and
- $A_S := {\sigma_S}^2$ is a stopping derivative for $\mathrm{Var}[X_T \mid \mathcal{F}_S]$ at S.

Itô's formula for stopping derivatives

Theorem Suppose a continuous adapted process $(X_t)_t$ has drift B_S and variance rate A_S at a finite stopping time S. Suppose also that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Then $(f(X_t))_t$ has drift D_S and variance rate C_S at S, where:

$$
\begin{aligned}
D_S &= f'(X_S)B_S + \frac{f''(X_S)}{2}A_S \\
C_S &= (f'(X_S))^2 A_S
\end{aligned}
$$

(With suitable tweaks to the definitions, the result generalises to cadlag adapted processes.)

A characterisation of continuous local martingales

Say that a process has zero drift if it has drift 0 at every finite stopping time.

Theorem A continuous adapted process has zero drift if and only if it is a local martingale.

Some questions

- Are finite stopping times dense? I.e., if $S < U$ a.s., does there exist T such that $S < T$ and $T < U$ a.s.?
- If a continuous process has drift 0 at all deterministic times then does it have drift 0 at all finite stopping times?
- In the fundamental theorem of calculus for stopping derivatives, is it enough to assume that $(B_t)_t$ and $(\sigma_t)_t$ are right-continuous at S?