Sheaves & Proofs

Alex Simpson

FMF, University of Ljubljana

IMFM, Ljubljana

alexksimpson.github.io

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Desired background

Some exposure to proof theory (sequent calculus) and semantics (structures, satisfaction, soundness, completeness) for classical predicate logic.

Not required background

- Intuitionistic logic (sequent calculus, Kripke models)
- Basic category theory (categories, functors, natural transformations)
- Basic point-set topology (topologies, covers)

Part 1

Coverage Semantics over Partial Orders

(The logic of localic toposes)

- Kripke semantics
- Cut-elimination via constructive completeness
- Coverage semantics
- The $\neg\neg$-dense coverage.

(Inverted) Kripke semantics

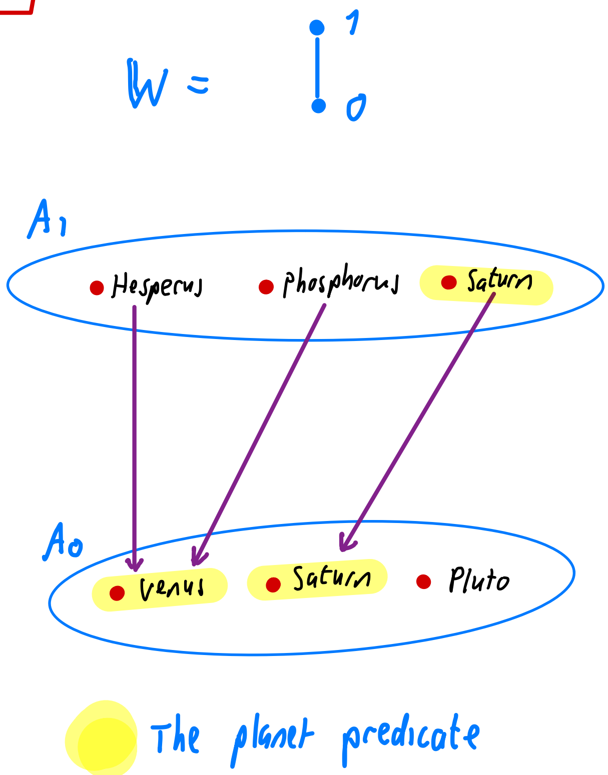
A Kripke model is given by

- A partial order $(\mathbb{W}, \leq)$ of worlds
- To every $w \in \mathbb{W}$ a set A_w
- Transition functions $(t_{vw} : A_w \to A_v)_{v \leq w}$

  satisfying $t_{uu} = id_{A_u}$; $u \leq v \leq w \Rightarrow t_{uw} = t_{uv} \circ t_{vw}$
- For every predicate P of arity k

  Subsets $P_w \subseteq (A_w)^k$ satisfying

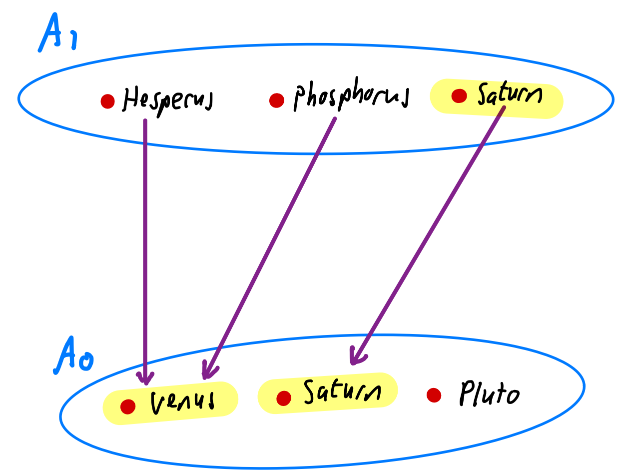
  $v \leq w \Rightarrow t_{vw}(P_w) \subseteq P_v$.



Category-theoretic formulation

A Kripke model is given by

- A partial order $(W, \leq)$ of worlds
- A functor $A : W^{op} \to \mathbf{Set}$
  (i.e., a presheaf $A \in \mathrm{PSh}(W)$)
- For every predicate P of arity k a subpresheaf $P \subseteq A^k$



The planet predicate

Notation

$$\begin{aligned} \text{Venus} &= t_{01}(\text{Hesperus}) && \text{(previous slide)} \\ &= A(0<1)(\text{Hesperus}) && \text{(this slide)} \\ &= \text{Hesperus} \cdot 0 && \text{(henceforth)} \end{aligned}$$

Kripke forcing relation

$$w \Vdash \phi$$

$w \Vdash P(a_1, \ldots, a_k) \iff (a_1, \ldots, a_k) \in P_w$

$w \Vdash a_1 = a_2 \iff a_1 = a_2$

$w \Vdash \phi \wedge \psi \iff w \Vdash \phi$ and $w \Vdash \psi$

$w \Vdash \phi \vee \psi \iff w \Vdash \phi$ or $w \Vdash \psi$

$w \Vdash \bot \iff$ false

$w \Vdash \exists x \phi \iff$ there exists $a \in A_w$ s.t. $w \Vdash \phi[x := a]$

$w \Vdash \phi \to \psi \iff$ for all $v \leq w$ $v \Vdash \phi \cdot v$ implies $v \Vdash \psi \cdot v$

$w \Vdash \forall x \phi \iff$ for all $v \leq w$ and $a \in A_v$ $v \Vdash (\phi \cdot v)[x := a]$

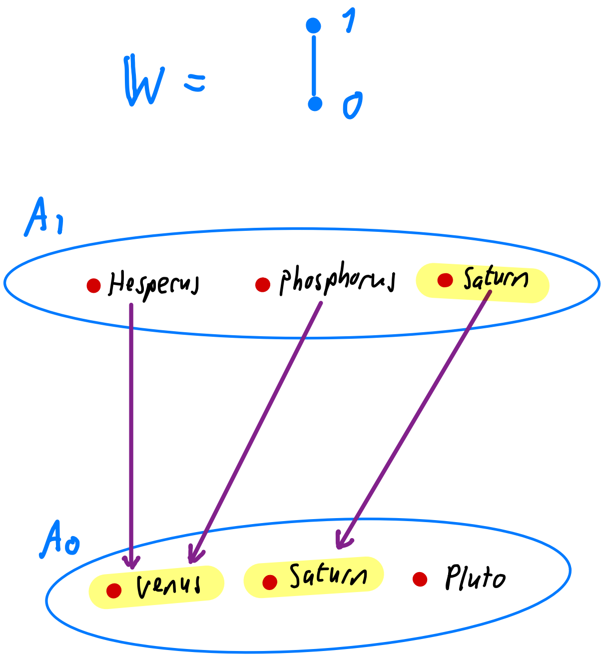
$1 \nVdash \text{Hesperus} = \text{Phosphorus}$

$0 \Vdash \text{Hesperus}\cdot 0 = \text{Phosphorus}\cdot 0$

$1 \nVdash \neg(\text{Hesperus} = \text{Phosphorus})$

$1 \nVdash H = P \vee \neg(H = P)$

$1 \Vdash \neg\neg(H = P)$



The planet predicate

$\neg\phi := \phi \to \bot$

$w \Vdash \neg\phi \iff \forall v \leq w .\ v \nVdash \phi\cdot v$

$w \Vdash \neg\neg\phi \iff \forall v \leq w\ \neg\neg\ \exists u \leq v\ \ u \Vdash \phi\cdot u$

Meta theorems for Kripke semantics

Monotonicity If $w \Vdash \phi$ and $v \le w$ then $v \Vdash \phi$. ✓

Soundness If ϕ is provable in intuitionistic predicate logic then for all Kripke models $(W, \dots)$ and $w \in W$, $w \Vdash \phi$.

Completeness If, for all Kripke models $(W, \dots)$ and $w \in W$, $w \Vdash \phi$ then ϕ is provable in intuitionistic predicate logic

Completeness is <u>not</u> provable in a constructive meta-theory .

To achieve constructive completeness, we need more general notion of model .

Coverage base

a.k.a. base for a Grothendieck topology on a poset

A coverage base on a poset $\mathbb{W}$ is a relation $\rhd$ relating pairs $C \rhd w$, where $w \in \mathbb{W}$ and $C \subseteq {\downarrow} w$, satisfying:

$${\downarrow} w := \{ v \in \mathbb{W} \mid v \leq w \}$$

$$\rhd \subseteq \sum_{w \in \mathbb{W}} \mathcal{P}({\downarrow} w)$$

Reflexivity $\quad \{w\} \rhd w$

Transitivity $\quad$ If $C \rhd w$ and, for all $v \in C$, $C_v \rhd v$

then $\bigcup_{v \in C} C_v \rhd w$.

Stability $\quad$ If $C \rhd w$ and $w' \leq w$ then there exists $D \rhd w'$ such that, for all $v' \in D$ there exists $v \in C$ such that $v' \leq v$.

Example coverage bases

① The <u>identity coverage</u> $\{w\} \rhd w$ on any poset $\mathbb{W}$

② For any topological space T, define

$$\mathbb{W} = \mathcal{O}(T)$$ (open sets)

$$v \leq w \;:\Leftrightarrow\; v \subseteq w$$

$$C \rhd w \;:\Leftrightarrow\; \bigcup C = w$$ (C covers w)

<u>Exercise</u> Prove that ② satisfies properties of a cover base!

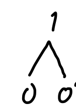
A $\boxed{\underline{\text{separated model}}}$ is given by

- A partial order $(W, \leq)$ of <u>worlds</u> with coverage base $\rhd$
- To every $w \in W$ a set A_w
- <u>Transition functions</u> $(t_{vw} : A_w \to A_v)_{v \leq w}$ s.t. $u \leq v \leq w \Rightarrow t_{uw} = t_{uv} \circ t_{vw}$
- $C \rhd w \Rightarrow \forall a, b \in A_w \; (\forall v \in C, \; t_{vw}(a) = t_{vw}(b)) \Rightarrow a = b$. ($\lhd$ - separated)
- For every predicate P of arity k subsets $P_w \subseteq (A_w)^k$ satisfying
  - $\forall \vec{a} \in A_w{}^k \; (\vec{a} \in P_w \Rightarrow \forall v \leq w, \; t_{vw}(\vec{a}) \in P_v)$
  - $C \rhd w \Rightarrow \forall \vec{a} \in A_w{}^k \; (\forall v \in C, \; t_{vw}(\vec{a}) \in P_v) \Rightarrow \vec{a} \in P_w$ (P is $\lhd$-closed)

Category-theoretic Formulation

A <u>separated model</u> is given by

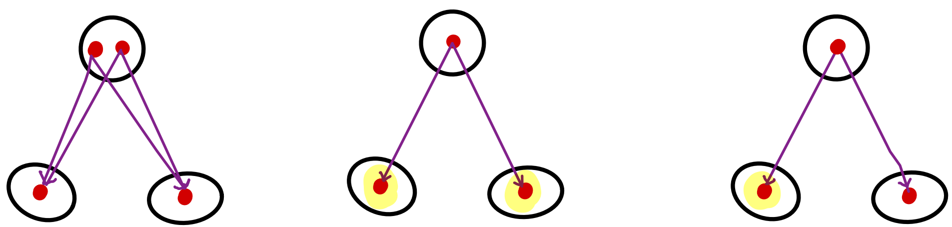
- A partial order $(\mathbb{W}, \leq)$ of worlds
- A <u>$\lhd$-separated</u> presheaf $A : \mathbb{W}^{op} \to \underline{Set}$
- For every predicate P of arity k, a <u>$\lhd$-closed</u> subpresheaf $P \subseteq A^k$

$\mathbb{W} =$ 

$\{0, 0'\} \rhd 1$

$\{0\} \not\rhd 1$

$\{0'\} \not\rhd 1$



not separated — not closed — a separated model

New forcing clauses

$w \Vdash \phi \vee \psi \iff$ there exists a cover $C \rhd w$ such that, for every $v \in C$, $v \Vdash \phi \cdot v$ or $v \Vdash \psi \cdot v$

$w \Vdash \bot \iff \emptyset \rhd w$

$w \Vdash \exists x\, \phi \iff$ there exists a cover $C \rhd w$ such that, for every $v \in C$, there exists $a \in \underline{A}_v$ s.t. $v \Vdash (\phi \cdot v)[x := a]$

N.B.

- The clauses for predicates, $=$, $\wedge$, $\to$, $\forall$ are as for Kripke semantics

For our example coverage bases

1. The identity coverage base $\{w\} \rhd w$ on any poset W

   Recovers Kripke semantics

2. The topological cover relation for $W = \mathcal{O}(T)$.

   Recovers topological semantics of intuitionistic logic

Meta theorems for coverage semantics

Monotonicity If $w \Vdash \phi$ and $v \leq w$ then $v \Vdash \phi \cdot v$

Cover property If $C \rhd w$ satisfies, for all $v \in C$, $v \Vdash \phi \cdot v$, then $w \Vdash \phi$.

Soundness If ϕ is provable in intuitionistic predicate logic then for all separated models $(W, \ldots)$ and $w \in W$, $w \Vdash \phi$.

Completeness If, for all separated models $(W, \ldots)$ and $w \in W$, $w \Vdash \phi$ then ϕ is provable in intuitionistic predicate logic

Completeness is now provable in a constructive meta-theory!

Intuitionistic sequent Calculus

Predicate symbols $P(x_1, \ldots, x_k)$ $(k \geq 0)$

No function symbols (in particular no constants) (for brevity of exposition)

Equality $t_1 = t_2$

[The only terms are variables, but we use $t, t_1, \ldots$ as meta-variables in places where arbitrary terms would be allowed in presence of function symbols.]

Sequent: $\Theta ; \Gamma \Rightarrow \psi$

where Θ is finite set of variables

Γ is finite set of Θ-formulas, ψ is Θ-formula

A Θ-formula is a formula whose free variables are contained in Θ.

Rules

$$\frac{}{\Theta ; \Gamma, P(\vec{t}) \Rightarrow P(\vec{t})}$$

$$\frac{\Theta ; \Gamma, \phi, \phi' \Rightarrow \psi}{\Theta ; \Gamma \ni \phi \wedge \phi' \Rightarrow \psi} \qquad \frac{\Theta ; \Gamma \Rightarrow \psi \quad \Theta ; \Gamma \Rightarrow \psi'}{\Theta ; \Gamma \Rightarrow \psi \wedge \psi'}$$

$$\frac{\Theta ; \Gamma, \phi \Rightarrow \psi \quad \Theta ; \Gamma, \phi' \Rightarrow \psi}{\Theta ; \Gamma \ni \phi \vee \phi' \Rightarrow \psi} \qquad \frac{\Theta ; \Gamma \Rightarrow \psi}{\Theta ; \Gamma \Rightarrow \psi \vee \psi'} \qquad \frac{\Theta ; \Gamma \Rightarrow \psi'}{\Theta ; \Gamma \Rightarrow \psi \vee \psi'}$$

$$\frac{}{\Theta ; \Gamma \ni \bot \Rightarrow \psi}$$

$$\frac{\Theta ; \Gamma, \phi' \Rightarrow \psi \quad \Theta ; \Gamma \Rightarrow \phi}{\Theta ; \Gamma \ni \phi \to \phi' \Rightarrow \psi} \qquad \frac{\Theta ; \Gamma, \psi \Rightarrow \psi'}{\Theta ; \Gamma \Rightarrow \psi \to \psi'}$$

$$\frac{\Theta \uplus x;\ \Gamma, \phi \Rightarrow \psi}{\Theta;\ \Gamma \ni \exists x\, \phi \Rightarrow \psi} \qquad \frac{\Theta;\ \Gamma \Rightarrow \psi[t/x]}{\Theta;\ \Gamma \Rightarrow \exists x\, \psi}$$

$$\frac{\Theta;\ \Gamma, \phi[t/x] \Rightarrow \psi}{\Theta;\ \Gamma \ni \forall x\, t \Rightarrow \psi} \qquad \frac{\Theta \uplus x;\ \Gamma \Rightarrow \psi}{\Theta;\ \Gamma \Rightarrow \forall x\, \psi}$$

$$\frac{\Theta;\ \Gamma \Rightarrow \psi[s/z]}{\Theta;\ \Gamma \ni s = t \Rightarrow \psi[t/z]} \qquad \frac{\Theta;\ \Gamma \Rightarrow \psi[t/z]}{\Theta;\ \Gamma, s = t \Rightarrow \psi[s/z]} \qquad \frac{}{\Theta;\ \Gamma \Rightarrow t = t}$$

$$\frac{\Theta;\ \Gamma \Rightarrow \phi \qquad \Theta;\ \Gamma, \phi \Rightarrow \psi}{\Theta;\ \Gamma \Rightarrow \psi}\ \text{cut}$$

Define $\Theta ; \Gamma \vdash \Psi$:$\Leftrightarrow$ the sequent $\Theta ; \Gamma \Rightarrow \Psi$ is provable
$\Theta ; \Gamma \vdash_{cf} \Psi$:$\Leftrightarrow$ the sequent $\Theta ; \Gamma \Rightarrow \Psi$ is cut-free provable.

Lemma (admissible rules with/without cut)

- (identity) $\Theta ; \Gamma, \Psi \vdash_{cf} \Psi$
- (weakening) $\Theta ; \Gamma \vdash \Psi \implies \Theta, \Theta' ; \Gamma, \Gamma' \vdash \Psi$

  $\Theta ; \Gamma \vdash_{cf} \Psi \implies \Theta, \Theta' ; \Gamma, \Gamma' \vdash_{cf} \Psi$

Proofs Straightforward inductions.

Theorem (Cut admissibility) $\Theta; \Gamma \vdash_{cf} \phi$ and $\Theta; \Gamma, \phi \vdash_{cf} \psi \implies \Theta; \Gamma \vdash_{cf} \psi$

We shall give a semantic proof, by establishing

Soundness $\Theta; \Gamma \vdash \psi \implies \Theta; \Gamma \vDash \psi$

Cut-free completeness $\Theta; \Gamma \vDash \psi \implies \Theta; \Gamma \vdash_{cf} \psi$

where $\vDash$ is semantic entailment w.r.t. separated models.

The proof will be carried out in a constructive meta-theory, so (in principle) it's possible to extract from the proof an algorithm that maps derivations with cuts to cut-free derivations.

Semantic entailment

$\Theta ; \Gamma \vDash \psi$

$:\Longleftrightarrow$

for every poset $\mathbb{W}$ and coverage base $\rhd$ on $\mathbb{W}$,

for every separated model $(A : \mathbb{W}^{op} \to \underline{\mathrm{Set}}, \{P_w\}_w)$,

$\forall \mathcal{M}$

for every $w \in \mathbb{W}$,

for every $\rho : \Theta \to A_w$

if, for every $\phi \in \Gamma$, $w \Vdash \phi[\rho]$,

then $w \Vdash \psi[\rho]$.

$\Theta ; \Gamma \vDash_{\mathcal{M}} \psi$

Proof of soundness: $\Theta; \Gamma \vdash \psi \implies \Theta; \Gamma \models \psi$

Consider any $\mathfrak{M}$, (i.e., $\mathbb{W}$, $\triangleright$ and separated model A, ...)

We prove by ind. on derivation of $\Theta; \Gamma \vdash \psi$ that $\Theta; \Gamma \models_{\mathfrak{M}} \psi$ holds

Exercise!

Proof of cut-free completeness: $\Theta ; \Gamma \vDash \psi \implies \Theta ; \Gamma \vdash_{cf} \psi$

In Part 1, we consider just the case of intuitionistic <u>propositional</u> logic

- All predicate symbols have arity 0.
  (Predicates of arity 0 are atomic propositions.)
- No variables, no quantifiers $\forall, \exists$
- No equality $\doteq$

We build a specific separated model $\mathcal{M}_{ctxt}$

The modal $\mathcal{M}_{ctxt}$ (propositional logic)

$W_{ctxt} := \{ \Gamma \mid \Gamma \text{ a finite set of formulas} \}$ (The set of contexts)

$\Gamma' \leq \Gamma \;:\Leftrightarrow\; \Gamma' \supseteq \Gamma$

$C \rhd_{ctxt} \Gamma \;:\Leftrightarrow\; \dfrac{C}{\Gamma}$ is a derivable context rule

$A_\Gamma := \emptyset$

$() \in P_\Gamma \;:\Leftrightarrow\; p \in \Gamma$. $(\Gamma \Vdash p \;:\Leftrightarrow\; p \in \Gamma)$

Context rules

$$\frac{\Gamma, \phi, \phi'}{\Gamma \ni \phi \wedge \phi'}$$

$$\frac{\Gamma, \phi \qquad \Gamma, \phi'}{\Gamma \ni \phi \vee \phi'}$$

$$\frac{}{\Gamma \ni \bot}$$

$$\frac{\Gamma, \phi'}{\Gamma \ni \phi \to \phi'} \; \Gamma \vdash_{cf} \phi$$

For any context Γ and set of contexts C,

$\dfrac{C}{\Gamma}$ is a derivable context rule if there exists a derivation tree of contexts with conclusion Γ, in which every leaf is in C.

Lemma Suppose $\dfrac{C}{\Gamma}$ is a derivable context rule.

- (weakening) For any Γ'',

$$\frac{\{\Gamma', \Gamma'' \mid \Gamma' \in C\}}{\Gamma, \Gamma''}$$

is a derivable context rule

The stability of $\rhd$ ctxt follows.

- (sequenting) For any ψ,

$$\frac{\{\Gamma' \Rightarrow \psi \mid \Gamma' \in C\}}{\Gamma \Rightarrow \psi}$$

is a cut-free derivable sequent rule.

Main lemma

$\mathcal{M}_{ctx}$ enjoys the following.

For any $\Gamma \in W_{ctx}$ and formula ϕ

(1) if $\phi \in \Gamma$ then $\Gamma \Vdash \phi$.

(2) if $\Gamma \Vdash \phi$ then $\Gamma \vdash_{cf} \phi$.

Proof of cut-free completeness: $\Gamma \vDash \psi \implies \Gamma \vdash_{cf} \psi$

Suppose $\Gamma \vDash \psi$.

In particular $\Gamma \vDash_{\mathcal{M}_{ctx},\Gamma} \psi$. $\circledast$

By ① of main lemma, for every $\phi \in \Gamma$, $\Gamma \Vdash \phi$.

By $\circledast$, $\Gamma \Vdash \psi$.

By ② of main lemma, $\Gamma \vdash_{cf} \psi$.

$\square$

Main lemma M_{ctx} enjoys the following.

For any $\Gamma \in W_{ctx}$ and formula ϕ

(1) if $\phi \in \Gamma$ then $\Gamma \Vdash \phi$.

(2) if $\Gamma \Vdash \phi$ then $\Gamma \vdash_{cf} \phi$.

Proof

(1) and (2) are proved simultaneously by induction on ϕ.

$\square$

Case $\phi \equiv \phi_1 \to \phi_2$

(1) $\phi_1 \to \phi_2 \in \Gamma \Rightarrow \Gamma \Vdash \phi_1 \to \phi_2$

Suppose $\Gamma' \supseteq \Gamma$ and $\Gamma' \Vdash \phi_1$

By i.h. (2) $\Gamma' \vdash_{cf} \phi_1$

By ○, $\{\Gamma', \phi_2\} \rhd_{ctxt} \Gamma'$

By i.h. (1), $\Gamma', \phi_2 \Vdash \phi_2$

By covering property $\Gamma' \Vdash \phi_2$.

Case $\phi \equiv \phi_1 \to \phi_2$

(2) $\Gamma \Vdash \phi_1 \to \phi_2 \implies \Gamma \vdash_{cf} \phi_1 \to \phi_2$

By i.h. (1) $\Gamma, \phi_1 \Vdash \phi_1$

By , $\Gamma, \phi_1 \Vdash \phi_2$

By i.h. (2) $\Gamma, \phi_1 \vdash_{cf} \phi_2$

Hence $\Gamma \vdash_{cf} \phi_1 \to \phi_2$ by $(\to R)$.

Case $\phi \equiv \phi_1 \vee \phi_2$

(1) $\phi_1 \vee \phi_2 \in \Gamma \Rightarrow \Gamma \Vdash \phi_1 \vee \phi_2$

We have $\{(\Gamma, \phi_1), (\Gamma, \phi_2)\} \triangleright_{ctx\vdash} \Gamma$

By i.h. (1), $\Gamma, \phi_1 \Vdash \phi_1$ and $\Gamma, \phi_2 \Vdash \phi_2$.

Hence $\Gamma \vdash \phi_1 \vee \phi_2$.

Case $\phi \equiv \phi_1 \vee \phi_2$

(2) $\Gamma \Vdash \phi_1 \vee \phi_2 \implies \Gamma \vdash_{cf} \phi_1 \vee \phi_2$

By ○, there is $C \rhd_{ctxt} \Gamma$ such that

for every $\Gamma' \in C$, $\Gamma' \Vdash \phi_1$ or $\Gamma' \Vdash \phi_2$.

So for every $\Gamma' \in C$, $\Gamma' \vdash_{cf} \phi_1$ or $\Gamma' \vdash_{cf} \phi_2$, by i.h (2).

So for every $\Gamma' \in C$, $\Gamma' \vdash_{cf} \phi_1 \vee \phi_2$, by $(\vee R)$ rules.

Since $\dfrac{C}{\Gamma}$ is admissible, $\Gamma \vdash_{cf} \phi_1 \vee \phi_2$, by sequenting property.

$\square$

Sieve

In a <u>coverage</u> (rather than coverage base) we only allow covers $C \rhd w$, where C is a <u>sieve</u> on w.

A set $C \subseteq W$ is a <u>sieve</u> on $w \in W$ if

- $C \subseteq \downarrow w$
- $C = \downarrow C$ (C is down closed)

If C is a sieve on w and $v \leq w$ then

$$C \cdot v := \{ u \in C \mid u \leq v \}$$

is a sieve on v.

$w \mapsto \mathrm{Sieves}(w)$ defines a presheaf $W^{op} \to \underline{\mathbf{Set}}$

Coverage

A <u>coverage</u> is a relation $C \rhd w$ between $w \in \mathbb{W}$ and <u>sieves</u> C on w satisfying

<u>Reflexivity</u> $\quad \downarrow w \rhd w$

<u>Transitivity</u> $\quad$ If $C \rhd w$ and S is a sieve on w satisfying

$$\forall v \in C, \quad S \cdot v \rhd v$$

then $S \rhd w$.

<u>Stability</u> $\quad$ If $C \rhd w$ and $v \leq w$ then $C \cdot v \rhd v$.

Proposition (Properties of coverages)

Let $\rhd$ be a coverage. Then $\rhd$ satisfies

Monotonicity If $C \rhd w$ and S is a sieve on w then

$$C \subseteq S \Rightarrow S \rhd w$$

Intersection If $C \rhd w$ and $C' \rhd w$ then $C \cap C' \rhd w$

Proof hints For monotonicity use reflexivity + transitivity.

For intersection use transitivity + stability.

Coverages vs. Coverage bases

Proposition

(1) If $\rhd$ is a coverage base then

$$C \rhd w \quad :\Leftrightarrow \quad \exists B \subseteq C,\ B \rhd w \text{ and } C = \downarrow B$$

is a coverage. We say $\rhd$ is the coverage <u>generated</u> by $\rhd$.

(2) Every coverage $\rhd$ is generated by the coverage base

$$B \rhd w \quad :\Leftrightarrow \quad \downarrow B \rhd w .$$

Proof

Exercise!

The $\neg\neg$-dense coverage

On any poset $\mathbb{W}$ define

$$C \rhd_{\neg\neg} w \quad :\Leftrightarrow \quad \begin{array}{l} C \text{ is a sieve on } w \text{ and} \\ \forall v \leq w \ \neg\neg \ \exists u \leq v \ u \in C. \end{array}$$

<u>Proposition</u> $\rhd_{\neg\neg}$ is a coverage.

<u>Proof</u> Exercise.

$\neg\neg$ Dense $\Rightarrow$ Classical

Proposition Using the coverage $\rhd_{\neg\neg}$, $\forall w \in \mathbb{W}$ $w \Vdash \neg\neg\phi \to \phi$

Proof Suppose $w \Vdash \neg\neg\phi$

We show $\{u \leq w \mid u \Vdash \phi\} \rhd_{\neg\neg} w$. Then $w \Vdash \phi$ by cover property.

$\{u \leq w \mid u \Vdash \phi\}$ is a sieve by monotonicity of $\Vdash$

Suppose $v \leq w$. We need to show $\neg\neg\, \exists u \leq v,\ u \Vdash \phi$.

Suppose for $\perp$ that $\neg\, \exists u \leq v,\ u \Vdash \phi$.

Then $v \Vdash \neg\phi$.

By $\circ$, $v \Vdash \perp$. So $\emptyset \rhd_{\neg\neg} v$. So $\forall v' \leq v$ $\neg\neg\, \exists u' \leq v', u' \in \emptyset$. I.e. $\perp$.

$\square$