A Tutorial on Sheaf Semantics

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Tutorial aimed at:

- Logicians
- Category theorists
- People interested in application areas
  - Nominal sets
  - Sheaves for contextuality
  - Sheaf models of probability
  - Sheaf models of type theory
- The LICS tourist

Part 1

Sheaf Semantics over Partial Orders

(The logic of localic toposes)

- Kripke semantics
- Topological sheaf semantics
- Coverage-based sheaf semantics over partial orders
- Dense sheaf semantics over partial orders

(Inverted) Kripke semantics

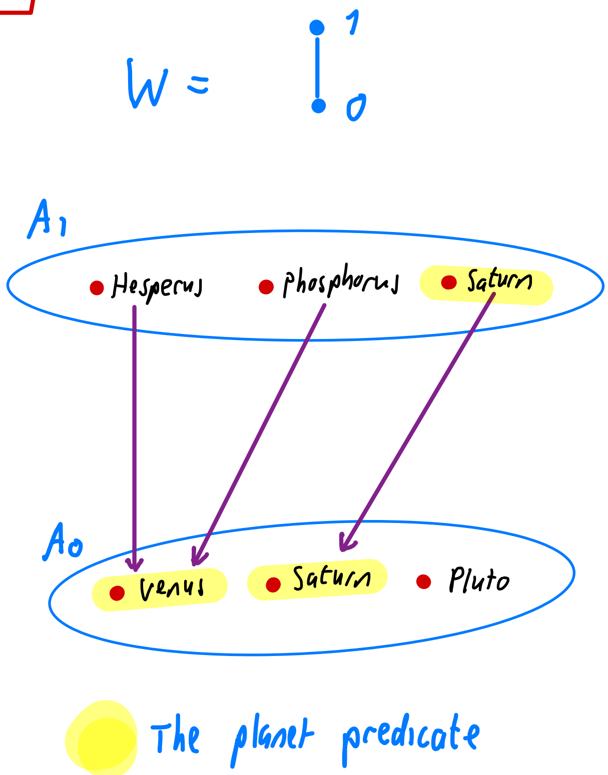
A <u>Kripke model</u> is given by

- A partial order $(W, \leq)$ of <u>worlds</u>
- To every $w \in W$ a set A_w
- <u>Transition functions</u> $(t_{wv} : A_w \to A_v)_{v \leq w}$

  satisfying $u \leq v \leq w \Rightarrow t_{wu} = t_{vu} \circ t_{wv}$
- For every predicate P of arity k

  Subsets $P_w \subseteq (A_w)^k$ satisfying

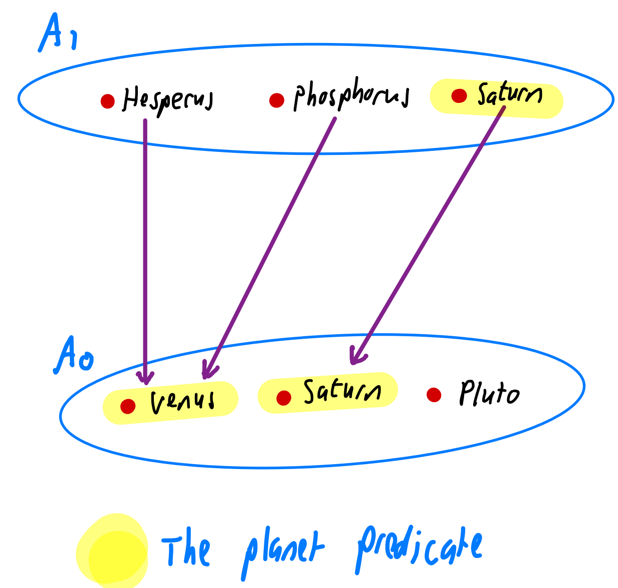
  $v \leq w \Rightarrow t_{wv}(P_w) \subseteq P_v$.



Category-theoretic formulation

A <u>Kripke model</u> is given by

- A partial order $(W, \leq)$ of <u>worlds</u>
- A functor $A : W^{op} \to \underline{\mathbf{Set}}$
  (i.e., a <u>presheaf</u> $A \in \mathrm{PSh}(W)$)
- For every predicate P of arity k a <u>subpresheaf</u> $P \subseteq A^k$



<u>Notation</u>

$\text{Venus} = t_{10}(\text{Hesperus})$ (previous slide)
$= A(0<1)(\text{Hesperus})$ (this slide)
$= \text{Hesperus} \cdot 0$ (henceforth)

Kripke forcing relation

$$w \Vdash \phi$$

$w \Vdash P(a_1, \dots, a_k) \iff (a_1, \dots, a_k) \in P_w$

$w \Vdash a_1 = a_2 \iff a_1 = a_2$

$w \Vdash \phi \wedge \psi \iff w \Vdash \phi$ and $w \Vdash \psi$

$w \Vdash \phi \vee \psi \iff w \Vdash \phi$ or $w \Vdash \psi$

$w \Vdash \bot \iff$ false

$w \Vdash \exists x\, \phi \iff$ there exists $a \in A_w$ s.t. $w \Vdash \phi[x := a]$

$w \Vdash \phi \to \psi \iff$ for all $v \leq w$ $\;v \Vdash \phi \cdot v$ implies $v \Vdash \psi \cdot v$

$w \Vdash \forall x\, \phi \iff$ for all $v \leq w$ and $a \in A_v$ $\;v \Vdash (\phi \cdot v)[x := a]$

$1 \nVdash \text{Hesperus} = \text{Phosphorus}$

$0 \Vdash \text{Hesperus}\cdot 0 = \text{Phosphorus}\cdot 0$

$1 \nVdash \neg(\text{Hesperus} = \text{Phosphorus})$

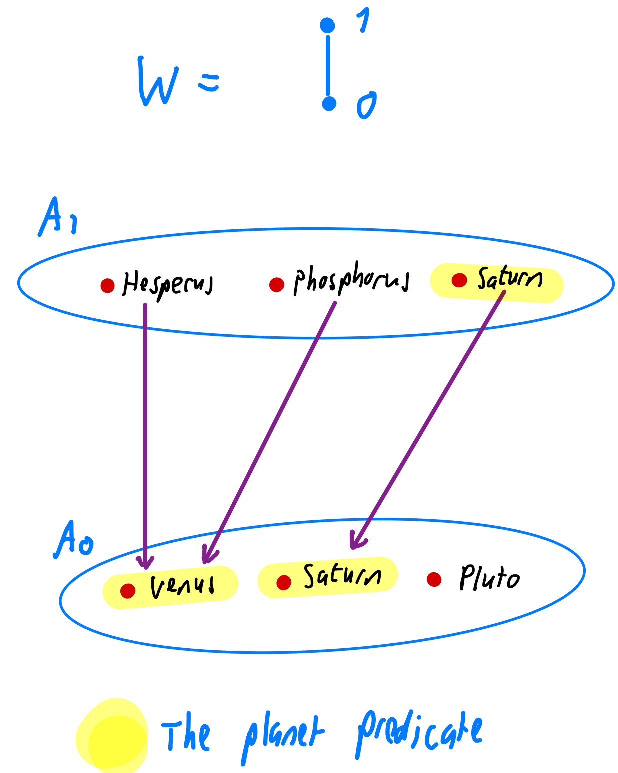
$1 \nVdash H = P \vee \neg(H = P)$

$1 \Vdash \neg\neg(H = P)$

$\neg\phi :\equiv \phi \to \bot$

$w \Vdash \neg\phi \iff \forall v \leq w \,.\, v \nVdash \phi\cdot v$

$w \Vdash \neg\neg\phi \iff \forall v \leq w \ \exists u \leq v \ \ u \Vdash \phi\cdot u$



Meta theorems for Kripke semantics

Monotonicity If $w \Vdash \phi$ and $v \leq w$ then $v \Vdash \phi \cdot v$

Soundness If ϕ is provable in intuitionistic predicate logic then for all Kripke models $(W, \ldots)$ and $w \in W$, $w \Vdash \phi$.

Completeness If, for all Kripke models $(W, \ldots)$ and $w \in W$, $w \Vdash \phi$ then ϕ is provable in intuitionistic predicate logic

(Classical meta-theory!)

Topological sheaf semantics

$\mathbb{W} = \mathcal{O}(T)$ (open subsets of a topological space T)

$v \leq w \iff v \subseteq w$

Say that $C \subseteq \mathcal{O}(u)$ is a <u>cover</u> of w (notation $C \rhd w$) if $\bigcup C = w$.

<u>Example cover</u> for $\mathbb{W} = \mathcal{O}(\mathbb{R})$

$$\left\{ (0, 2/3), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{2}{3}, \frac{4}{5}\right), \left(\frac{3}{4}, \frac{5}{6}\right), \dots \right\} = \left\{ \left(\frac{n}{n+1}, \frac{n+2}{n+3}\right) \,\middle|\, n \in \mathbb{N} \right\}$$

is a cover of $(0,1)$

Given a presheaf $A : \mathbb{W}^{op} \to \underline{\mathrm{Set}}$ (where $\mathbb{W} = \mathcal{O}(T)$).

- A <u>matching family</u> for a cover $C \rhd w$ is a family $(a_v \in A_v)_{v \in C}$ such that $\forall u, v \in C \quad a_u \cdot (u \wedge v) = a_v \cdot (u \wedge v)$

- An <u>amalgamation</u> for a (necessarily matching) family $(a_v \in A_v)_{v \in C}$ is an element $a_w \in A_w$ s.t. $\forall v \in C \quad a_v = a_w \cdot v$.

- The presheaf A is a <u>sheaf</u> if every matching family has a unique amalgamation.

Example sheaf

$$\underline{R}_w := \{ f : w \to \mathbb{R} \mid f \text{ continuous} \}$$

$$f \cdot v := f|_v \qquad\qquad (f \in \underline{R}_w,\ v \le w)$$

New forcing clauses

$w \Vdash \phi \vee \psi \iff$ there exists a cover $C \rhd w$ such that,
for every $v \in C$, $v \Vdash \phi \cdot v$ or $v \Vdash \psi \cdot v$

$w \Vdash \bot \iff w = \emptyset \quad (\iff \emptyset \rhd w)$

$w \Vdash \exists x\, \phi \iff$ there exists a cover $C \rhd w$ such that,
for every $v \in C$,
there exists $a \in \underline{A}_v$ s.t. $v \Vdash (\phi \cdot v)[x := a]$

N.B.

- The clauses for predicates, $=$, $\wedge$, $\rightarrow$, $\forall$ are as before,
- Variables range over a sheaf $\underline{A}$

A sentence ϕ is <u>valid</u> in sheaves over T $(\mathrm{Sh}(T) \vDash \phi)$ if

$$\forall w \in \mathcal{O}(T) \quad w \Vdash \phi .$$

<u>Example</u> $\mathrm{Sh}(\mathbb{R}) \nVdash \forall x \;\; x \leq 0 \vee x \geq 0$

(variables interpreted in the sheaf $\underline{\mathbb{R}}$).

<u>Proof</u>

Suppose for contradiction that $\mathbb{R} \Vdash \forall x \;\; x \leq 0 \vee x \geq 0$

Then $\mathbb{R} \Vdash id \leq 0 \vee id \geq 0$ $\quad$ ($id : \mathbb{R} \to \mathbb{R}$ the identity function)

There is a cover $C \triangleright \mathbb{R}$ such that, for every $U \in C$, $U \Vdash id \leq 0$ or $U \Vdash id \geq 0$

i.e., $U \subseteq (-\infty, 0]$ or $U \subseteq [0, \infty)$

Since $\bigcup C = \mathbb{R}$, there exists $U_0 \in C$ s.t. $0 \in U_0$.

Since U_0 is open $(-\varepsilon, \varepsilon) \subseteq U_0$ for some $\varepsilon > 0$, contradicting $U \subseteq (-\infty, 0]$ or $U \subseteq [0, \infty)$.

$\square$

Example $Sh(T) \models \forall x \, \forall \varepsilon > 0 \quad x > 0 \vee x < \varepsilon$

Proof Consider any $W \in \mathcal{O}(T)$,

$f : W \to \mathbb{R}$ continuous,

$\varepsilon : W \to (0, \infty)$ continuous. We show $W \Vdash f > 0 \vee f < \varepsilon$.

Define $V_1 := f^{-1}(0, \infty)$. $V_1 \subseteq W$ is open because f continuous.

Define $V_2 := \{ z \in W \mid f(z) < \varepsilon(z) \}$.

$V_2 \subseteq W$ is open because $< \subseteq \mathbb{R} \times \mathbb{R}$ is open, f, ε are continuous and $V_2 = (f, \varepsilon)^{-1}(<)$.

By definition $V_1 \Vdash f > 0$ and $V_2 \Vdash f < \varepsilon$

Moreover $V_1 \cup V_2 = W$ because for all $z \in W$ $f(z) > 0$ or $f(z) \leq 0 < \varepsilon$.

So $W \Vdash f > 0 \vee f < \varepsilon$. $\square$

Coverage (base)

a.k.a. base for a Grothendieck topology on a poset

A <u>coverage (base)</u> on a poset $\mathbb{W}$ is a relation of the form $C \rhd w$, where $w \in \mathbb{W}$ and $C \subseteq {\downarrow} w$, satisfying: $({\downarrow} w := \{v \in \mathbb{W} \mid v \leq w\})$

<u>Reflexivity</u> $\{w\} \rhd w$

<u>Transitivity</u> If $C \rhd w$ and, for all $v \in C$, $C_v \rhd v$

then $\bigcup_{v \in C} C_v \rhd w$.

<u>Stability</u> If $C \rhd w$ and $w' \leq w$ then there exists $D \rhd w'$ such that, for all $v' \in D$ there exists $v \in C$ such that $v' \leq v$.

Sheaf for a coverage

Given a presheaf $A : W^{op} \to \underline{Set}$ and coverage $\rhd$ on W.

- A <u>matching family</u> for a cover $C \rhd W$ is a family $(a_v \in A_v)_{v \in C}$ such that $\forall v, v' \in C$ $\forall u \in {\downarrow} v \cap {\downarrow} v'$ $a_v \cdot u = a_{v'} \cdot u$

- An <u>amalgamation</u> for a (necessarily matching) family $(a_v \in A_v)_{v \in C}$ is an element $a_w \in A_w$ s.t. $\forall v \in C$ $a_v = a_w \cdot v$.

- The presheaf A is a <u>sheaf</u> if every matching family has a unique amalgamation.

Forcing w.r.t. a coverage

$w \Vdash \phi \vee \psi \iff$
there exists a cover $C \rhd w$ such that,
for every $v \in C$, $v \Vdash \phi \cdot v$ or $v \Vdash \psi \cdot v$

$w \Vdash \bot \iff \emptyset \rhd w$

$w \Vdash \exists x\, \phi \iff$ there exists a cover $C \rhd w$ such that,
for every $v \in C$,
there exists $a \in \underline{A}_v$ s.t. $v \Vdash (\phi \cdot v)[x := a]$

Meta theorems for sheaf semantics

Monotonicity If $w \Vdash \phi$ and $v \leq w$ then $v \Vdash \phi \cdot v$

Sheaf property If $C \rhd W$ satisfies, for all $v \in C$, $v \Vdash \phi \cdot v$ then $w \Vdash \phi$.

Soundness If ϕ is provable in intuitionistic predicate logic then for all Kripke models $(W, \ldots)$ and $w \in W$, $w \Vdash \phi$.

Completeness If, for all Kripke models $(W, \ldots)$ and $w \in W$, $w \Vdash \phi$ then ϕ is provable in intuitionistic predicate logic

(Constructive meta-theory !?)

Example Coverages

(1) The topological cover relation for $\mathbb{W} = \mathcal{O}(T)$.

Recovers topological sheaf semantics

(2) The identity coverage $\{w\} \rhd w$ on any poset $\mathbb{W}$

Recovers Kripke semantics

(3) The dense coverage on any poset $\mathbb{W}$

$$C \rhd w \iff \text{for any } u \leq w \ \exists v \in C \text{ s.t. } {\downarrow} u \cap {\downarrow} v \neq \emptyset .$$

Dense $\Rightarrow$ Classical

Proposition

$$Sh_{den}(\mathbb{W}) \models \neg\neg\phi \to \phi$$

Proof

Suppose $w \Vdash \neg\neg\phi$

For the dense coverage, $\forall C, v \quad C \rhd v \Rightarrow C \neq \emptyset$

Thus $\neg\neg\phi$ gets its Kripke interpretation

Hence $\forall v \leq w \; \exists u \leq v \quad u \Vdash \phi$

That is $\{u \leq w \mid u \Vdash \phi\} \rhd w$ in the dense coverage

So, by the sheaf property of forcing, $w \Vdash \phi$.

$\square$

Let $\lhd$ be a coverage on a poset $\mathbb{W}$.

A <u>$\lhd$-ideal</u> (J-ideal) is a subset $I \subseteq \mathbb{W}$ such that

- $I = \downarrow I$ (down-closure)
- For any $C \rhd w$, $C \subseteq I \Rightarrow w \in I$ ($\lhd$-closure)

The set $\lhd$-Idl of $\lhd$-ideals partially ordered by $\subseteq$ is a complete Heyting algebra

Every CHA arises in this way (for some $\mathbb{W}$ and $\lhd$).

$$\boxed{\text{Sheaf semantics on posets} \quad \sim \quad \text{Heyting-valued semantics}}$$

When $\lhd$ is the dense coverage, $\lhd$-Idl is a complete Boolean algebra.

Sheaf semantics on posets dense coverages $\sim$ Boolean-valued semantics

Sheaf semantics for the dense coverage corresponds to Cohen-style forcing.

For any poset $\mathbb{W}$,

$$\mathrm{Sh}_{den}(\mathbb{W}) \models AC$$

Part 2

Sheaf Semantics over Categories

(The logic of Grothendieck toposes)

- Presheaf semantics
- Dense coverages
- (Coverage based) sheaf semantics
- Atomic coverages

Presheaf semantics

$$\mathbb{C} = \begin{tikzcd} 0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & 1 \end{tikzcd}$$

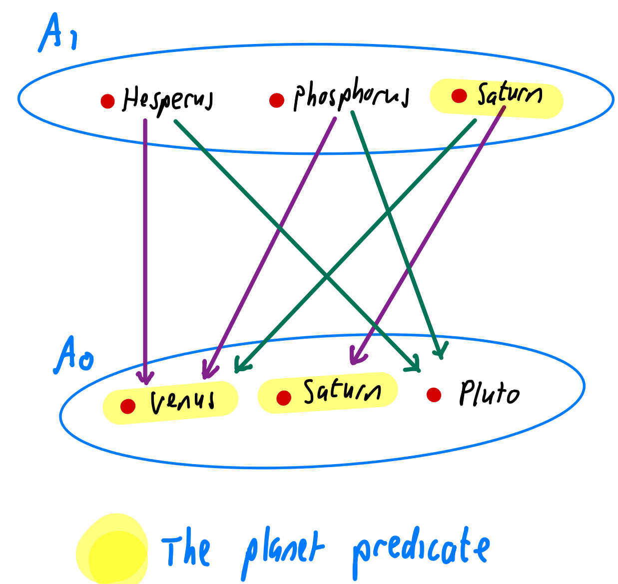
A **Presheaf model** is given by

- A (small) category $\mathbb{C}$ of **worlds**
- To every $X \in \mathbb{C}$ a set A_X
- **Transition functions** $(t_f : A_X \to A_Y)_{Y \xrightarrow{f} X}$

  satisfying $Z \xrightarrow{g} Y \xrightarrow{f} X \Rightarrow t_{f \circ g} = t_g \circ t_f$
- For every predicate P of arity K

  Subsets $P_w \subseteq (A_w)^K$ satisfying

  $Y \xrightarrow{f} X \Rightarrow t_f(P_X) \subseteq P_Y$

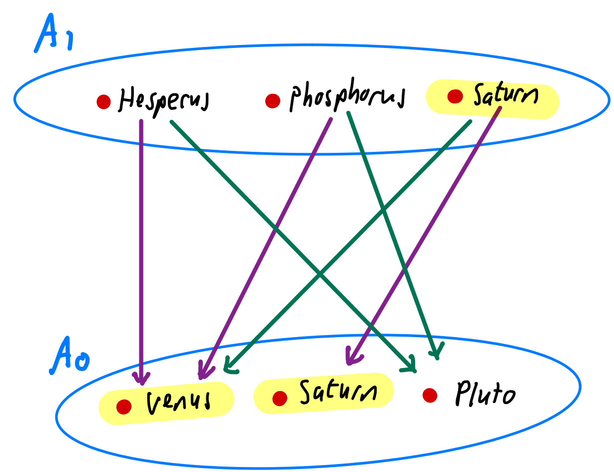


Category-theoretic formulation

A **Presheaf model** is given by

- A small category $\mathbb{C}$ of worlds
- A functor $A : \mathbb{C}^{op} \to \mathbf{Set}$
  (i.e., a presheaf $A \in \mathrm{PSh}(\mathbb{C})$)
- For every predicate P of arity k a subpresheaf $R \subseteq A^k$

$$\mathbb{C} = \begin{tikzcd} 0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & 1 \end{tikzcd}$$



The planet predicate

Notation

$$\begin{aligned} \text{Venus} &= t_{f_0}(\text{Hesperus}) && \text{(previous slide)} \\ &= A(f_0)(\text{Hesperus}) && \text{(this slide)} \\ &= \text{Hesperus} \cdot f_0 && \text{(henceforth)} \end{aligned}$$

Presheaf forcing relation

$X \Vdash \phi$

$X \Vdash P(a_1,\ldots,a_k) \iff (a_1,\ldots,a_k) \in P_X$

$X \Vdash a_1 = a_2 \iff a_1 = a_2$

$X \Vdash \phi \wedge \psi \iff X \Vdash \phi$ and $X \Vdash \psi$

$X \Vdash \phi \vee \psi \iff X \Vdash \phi$ or $X \Vdash \psi$

$X \Vdash \bot \iff$ false

$X \Vdash \exists x\, \phi \iff$ there exists $a \in A_X$ s.t. $X \Vdash \phi[x := a]$

$X \Vdash \phi \to \psi \iff$ for all $Y \xrightarrow{f} X$ $Y \Vdash \phi \cdot f$ implies $Y \Vdash \psi \cdot f$

$X \Vdash \forall x\, \phi \iff$ for all $Y \xrightarrow{f} X$ and $a \in A_Y$ $Y \Vdash (\phi \cdot f)[x := a]$

Meta theorems for Presheaf Semantics

Monotonicity If $X \Vdash \phi$ and $Y \xrightarrow{f} X$ then $Y \Vdash \phi \cdot f$

Soundness If ϕ is provable in intuitionistic predicate logic then for all presheaf models $(\mathbb{C}, \dots)$ and $X \in \mathbb{C}$, $X \Vdash \phi$.

Completeness If, for all presheaf models $(\mathbb{C}, \dots)$ and $X \in \mathbb{C}$, $X \Vdash \phi$ then ϕ is provable in intuitionistic predicate logic

(Classical meta-theory!)

Coverage (base)

a.k.a. base for a Grothendieck topology on a category

A coverage (base) on a category $\mathbb{C}$ is a relation of the form $C \rhd X$, where $X \in \mathbb{C}$ and $C \subseteq \bigcup_{Y \in \mathbb{C}} \mathbb{C}(Y, X)$

Reflexivity $\quad \{X \xrightarrow{id_X} X\} \rhd X$

Transitivity $\quad$ If $C \rhd X$ and, for all $Y \xrightarrow{f} X \in C$, $C_f \rhd Y$

then $\{Z \xrightarrow{g \circ f} X \mid Y \xrightarrow{f} X \in C,\ Z \xrightarrow{g} Y \in C_f\} \rhd X$

Stability $\quad$ If $C \rhd X$ and $X' \xrightarrow{f} X$ then there exists $D \rhd X'$ such that, for all $Y' \xrightarrow{g'} X' \in D$ there exist $Y \xrightarrow{g} X \in C$ and $Y' \xrightarrow{f'} X'$ such that

$$\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "f"'] & X
\end{tikzcd}$$

Sheaf for a coverage

Given a presheaf $A : \mathbb{C}^{op} \to \underline{\mathrm{Set}}$ and coverage $\rhd$ on $\mathbb{C}$.

- A <u>matching family</u> for a cover $C \rhd X$ is a family $(a_f \in A_Y)_{Y \xrightarrow{f} X \in C}$ such that $\forall\, Y \xrightarrow{f} X,\ Y' \xrightarrow{f'} X \in C,\ \forall$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "g'"'] & Y \arrow[d, "f"] \\ Y' \arrow[r, "f'"'] & X \end{tikzcd}$$

in $\mathbb{C}$, $a_f \cdot g = a_{f'} \cdot g'$.

- An <u>amalgamation</u> for a (necessarily matching) family $(a_f \in A_Y)_{Y \xrightarrow{f} X \in C}$ is an element $a \in AX$ s.t. $\forall\, Y \xrightarrow{f} X \in C,\ a_f = a \cdot f$.

- The presheaf A is a <u>sheaf</u> if every matching family has a unique amalgamation.

$$\boxed{\text{Forcing w.r.t. a coverage}}$$

$X \Vdash \phi \vee \psi \iff$

there exists a cover $C \rhd X$ such that,

for every $Y \xrightarrow{f} X \in C$, $Y \Vdash \phi \cdot f$ or $Y \Vdash \psi \cdot f$

$X \Vdash \bot \iff \emptyset \rhd X$

$X \Vdash \exists x\, \phi \iff$ there exists a cover $C \rhd X$ such that,

for every $Y \xrightarrow{f} X \in C$

there exists $a \in \underline{A}_Y$ s.t. $Y \Vdash (\phi \cdot f)[x := a]$

Meta theorems for sheaf semantics

Monotonicity If $X \Vdash \phi$ and $Y \xrightarrow{f} X$ then $Y \Vdash \phi \cdot f$

Sheaf property If $C \rhd X$ satisfies, for all $Y \xrightarrow{f} X \in C$, $Y \Vdash \phi \cdot f$
then $X \Vdash \phi$.

Soundness If ϕ is provable in intuitionistic predicate logic
then for all sheaf models $(\mathbb{C}, \rhd, \ldots)$ and $X \in \mathbb{C}$, $X \Vdash \phi$.

Completeness If, for all sheaf models $(\mathbb{C}, \rhd, \ldots)$ and $X \in \mathbb{C}$, $X \Vdash \phi$
then ϕ is provable in intuitionistic predicate logic

(Constructive meta-theory !?)

Dense coverage

For any small category $\mathbb{C}$, the <u>dense coverage base</u>

$$C \rhd X \iff \text{for all } Z \xrightarrow{f} X \text{ there exists } Y \xrightarrow{g} X \in C$$

such that the cospan $Z \xrightarrow{f} X \xleftarrow{g} Y$ completes to a commuting square in $\mathbb{C}$

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

Sheaf semantics for the dense coverage is <u>classical</u> $\quad X \Vdash \neg\neg\phi \to \phi$

Atomic coverage

Suppose $\mathbb{C}$ satisfies the following <u>coconfluence condition</u> (right Ore condition)

every cospan $Z \xrightarrow{f} X \xleftarrow{g} Y$ completes to a commuting square

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

Then

$$C \triangleright X \iff C = \text{a singleton } \{Y \xrightarrow{g} X\}$$

is a coverage base. The <u>atomic coverage base</u>

$$\boxed{\text{atomic} = \text{dense}}$$

A subset $S \subseteq \bigcup_{Y \in \mathbb{C}} \mathbb{C}(Y, X)$ is a <u>sieve</u> if (cf. ideal in a ring, down-closed set in a poset)

$$Y \xrightarrow{f} X \in S \text{ and } Z \xrightarrow{g} Y \in \mathbb{C} \Rightarrow Z \xrightarrow{g \circ f} X \in S$$

Given a coverage base $\rhd$ define

$$S \rhd^* X \iff S \text{ a sieve and } \exists C \subseteq S \;\; C \rhd X$$

If $\mathbb{C}$ is coconfluent then

$$\boxed{S \rhd^*_{den} X \iff S \rhd^*_{at} X \iff S \text{ a sieve and } S \neq \emptyset}$$

Dependent choice (DC)

$$\left(\forall x : A \;\; \exists y : A \;\; R(x,y)\right) \to$$
$$\forall x : A \;\; \exists s : A^{\mathbb{N}} \;\; s_0 = x \;\wedge\; \forall n : \mathbb{N} \;\; R(s_n, s_{n+1})$$

> **Proposition** If $\mathbb{C}$ is coconfluent and every ω^{op}-chain in $\mathbb{C}$ has a cone*
> then $Sh_{at}(\mathbb{C}) \models DC$

* For every $\mathbb{C}$ diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X_4 \arrow[r] & X_3 \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & X_0 \\
X_\omega \arrow[ur] \arrow[urr] \arrow[urrr] \arrow[urrrr] \arrow[urrrrr] & & & & &
\end{tikzcd}$$

then exists $(X_\omega \to X_n)_{n \in \mathbb{N}}$ such that all triangles commute

Proof Suppose $X_0 \Vdash \forall x{:}A\ \exists y{:}A\ R(x,y)$ and $a_0 \in A(X)$

Then $s := (a_n \cdot q_n)_{n<\omega} \in A^\omega(X_\omega)$ satisfies $X_\omega \Vdash s_0 = q_0 \cdot a_0 \wedge \forall n{:}N\ R(a_n, a_{n+1})$

$$
\begin{tikzcd}
X_\omega \arrow[r, "q_n"'] \arrow[ddr, "q_2"] \arrow[dddr, "q_1"] \arrow[ddddr, "q_0"'] & X_n \arrow[r, "a_n"] & A \\
\vdots & \vdots \arrow[d] & \\
 & X_2 \arrow[r, "a_2"] \arrow[d, "p_1"'] & A \\
 & X_1 \arrow[r, "a_1"] \arrow[d, "p_0"'] & A \\
 & X_0 \arrow[r, "a_0"'] & A
\end{tikzcd}
$$

$X_n \Vdash R(a_{n-1} \cdot p_{n-1}, a_n)$

...

$X_n \Vdash R(a_0 \cdot p_0 \cdot \cdots \cdot p_{n-1}, a_1 \cdot p_1 \cdot \cdots \cdot p_{n-1})$

$X_2 \Vdash R(a_1 \cdot p_1, a_2)$

$X_2 \Vdash R(a_0 \cdot p_0 \cdot p_1, a_1 \cdot p_1)$

$X_1 \Vdash R(a_0 \cdot p_0, a_1)$

for all n

$X_\omega \xrightarrow{a_n \cdot q_n} A$

$X_\omega \Vdash R(a_n \cdot q_n, a_{n+1} \cdot q_{n+1})$

The proposition can be used to show that atomic toposes of relevance to probabilistic semantics validate DC.

- The topos of probability sheaves
  Equivalence and conditional independence in atomic sheaf logic, S., LICS 2024
- The topos of enhanced measurable sheaves
  A nominal approach to probabilistic separation logic,
  Li, Ahmed, Aytac, Holtzen, Johnson-Freyd, LICS 2024.

Other applications of sheaf semantics

- Freyd's topos refuting AC
- Sheaf models of type theory
- The topological topos (Johnstone)
- The random topos (S.)
- Grothendieck toposes in mathematics
  - Zariski topos (SGA)
  - Condensed sets (Clausen, Scholze)
- etc.

Literature

The literature I know on sheaf semantics approaches it via first understanding toposes and their logic, requiring substantial category theory.

My favourite book that takes this approach is

- Mac Lane & Moerdijk *Sheaves in Geometry and Logic*

A more gently paced presentation (with less content) can be found in

- Goldblatt *Topoi*