Equivalence and Conditional Independence in Atomic Sheaf Logic

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Aim of talk

Logical reasoning principles for probabilistic relations:

$X = Y$	X and Y are almost surely equal
$X \sim Y$	X and Y are identically distributed
$X \perp Y$	X and Y are independent
$X \perp Y \mid Z$	X and Y are conditionally independent given Z

Example expressible property:

$$X \perp Y \ \wedge \ X \sim Y$$

says that X and Y are independent and identically distributed (iid).

Also interested in non-probabilistic interpretations of the same primitives.

Nondeterministic variables

A nondeterministic variable valued in a set A is function

$$X : \Omega \to A$$

where Ω is a finite nonempty sample set.

Nondeterministic variables X, Y are equiextensive ($X \bowtie Y$) if they have the same images:,

$$X(\Omega) = Y(\Omega) \quad .$$

(Conditional) independence of nondeterministic variables

Nondeterministic variables X, Y are independent $(X \perp\!\!\!\perp Y)$ if

$$\forall a, b \in A.\ \Diamond(X = a) \wedge \Diamond(Y = b) \ \rightarrow\ \Diamond(X = a \wedge Y = b)$$

X, Y are conditionally independent given Z $(X \perp\!\!\!\perp Y \mid Z)$ if

$$\forall a, b, c \in A.\ \Diamond(X = a \wedge Z = c) \ \wedge\ \Diamond(Y = b \wedge Z = c) \\ \rightarrow \Diamond(X = a \wedge Y = b \wedge Z = c)$$

Logical formulas

$$\Phi \;::=\; x = y \mid x \sim y \mid x \perp y \mid x \perp y \,|\, z \mid$$
$$\Phi \wedge \Phi \mid \neg\Phi \mid \exists x\ \Phi$$

The atomic formulas (first row) are: equality, equivalence, independence and conditional independence.

(The paper has multisorted variables and atomic formulas involving vectors of variables.)

The semantics of a formula $\Phi(x_1, \ldots, x_k)$ is defined via a forcing relation of the form

$$\Omega \Vdash_{\underline{\rho}} \Phi$$

where $\underline{\rho} \colon \{x_1, \ldots, x_k\} \to (\Omega \to A)$.

Variables are interpreted as nondeterministic variables.

Semantics of atomic formulas

$$\Omega \Vdash_{\underline{\rho}} x = y \iff \underline{\rho}(x) = \underline{\rho}(y)$$

$$\Omega \Vdash_{\underline{\rho}} x \sim y \iff \underline{\rho}(x) \bowtie \underline{\rho}(y)$$

$$\Omega \Vdash_{\underline{\rho}} x \perp y \iff \underline{\rho}(x) \perp\!\!\!\perp \underline{\rho}(y)$$

$$\Omega \Vdash_{\underline{\rho}} x \perp y \mid z \iff \underline{\rho}(x) \perp\!\!\!\perp \underline{\rho}(y) \mid \underline{\rho}(z)$$

Semantics of logical formulas

$$\Omega \Vdash_{\underline{\rho}} \Phi \wedge \Psi \iff \Omega \Vdash_{\underline{\rho}} \Phi \text{ and } \Omega \Vdash_{\underline{\rho}} \Psi$$

$$\Omega \Vdash_{\underline{\rho}} \neg\Phi \iff \Omega \nVdash_{\underline{\rho}} \Phi$$

$$\Omega \Vdash_{\underline{\rho}} \exists x.\, \Phi \iff \exists q \colon \Omega' \twoheadrightarrow \Omega.\, \exists X \colon \Omega' \to A.\ \Omega' \Vdash_{\underline{\rho'}[x:=X]} \Phi$$

$$\text{where } \underline{\rho'} \colon z \mapsto \underline{\rho}(z) \circ q$$

Relationship to independence logic

Variable assignments $\{x_1, \ldots, x_k\} \to (\Omega \to A)$ correspond to the multiteams (Durand et. al. 2017) of (in)dependence logic (Väänänen 2007, Grädel & Väänänen 2013).

The semantic clauses for atomic formulas, conjunction and the existential quantifier coincide with corresponding clauses in independence logic (under the lax semantics of $\exists$).

In independence logic, negation is usually restricted to atomic formulas and its semantics is defined differently. There are also semantic clauses for $\vee, \to, \forall$.

Independence logic is an exotic logic (e.g., disjunction is not idempotent) with characteristics that make it challenging to use as a framework for reasoning about independence (e.g., $\forall x \forall y.\ x \perp y$ is validated).

With our forcing relation, the logic is not exotic.

Theorem Every theorem of classical first-order logic is forced.

Accordingly, semantic clauses for $\vee, \to, \forall$ can be derived.

We obtain a classical logic for reasoning about equality, equivalence and conditional independence.

Axioms for conditional independence

The expected axioms are validated, giving a practical (cf. Dawid, Pearl, ...) axiomatisation of conditional independence.

- $x \perp w \mid w$
- $x \perp y \mid w \;\rightarrow\; y \perp x \mid w$
- $x \perp y, z \mid w \;\rightarrow\; x \perp y \mid w$
- $x \perp y, z \mid w \;\rightarrow\; x \perp y \mid z, w$
- $x \perp y \mid z, w \;\wedge\; x \perp z \mid w \;\rightarrow\; x \perp y, z \mid w$

Transfer principle

$$\exists x \quad x \sim x' \;\rightarrow\; \exists y' \; x, y \sim x', y'$$

Independence principle

$$\exists x \quad x \sim y \;\wedge\; x \perp z$$

Invariance principle

$$x \sim y \;\wedge\; \Phi(x) \;\rightarrow\; \Phi(y)$$

($\Phi(x)$ has at most one free variable)

Category-theoretic perspective

$\mathbb{S}\mathrm{ur}$ = category of finite nonempty sets and surjective functions.

Every set A, defines a presheaf $A^{(-)} \colon \mathbb{S}\mathrm{ur}^{\mathrm{op}} \to \mathbf{Set}$.

We have subpresheaves

$$
\begin{aligned}
(-) \sim (-) &\subseteq A^{(-)} \times A^{(-)} \\
(-) \perp\!\!\!\perp (-) &\subseteq A^{(-)} \times B^{(-)} \\
(-) \perp\!\!\!\perp (-) \,|\, (-) &\subseteq A^{(-)} \times B^{(-)} \times C^{(-)}
\end{aligned}
$$

$\mathbb{S}\mathrm{ur}$ is a coconfluent category, thus $\mathbb{S}\mathrm{ur}$ carries the atomic Grothendieck topology.

For every set A, the presheaf $A^{(-)}$ is an atomic sheaf.

The subpresheaves above are in fact subsheaves.

Our forcing relation is sheaf semantics in $\mathrm{Sh}_{\mathrm{at}}(\mathbb{S}\mathrm{ur})$.

The general story

Let $\mathbb{C}$ be any small coconfluent category.

Every sheaf $\underline{A}$ in $\mathrm{Sh}_{\mathrm{at}}(\mathbb{C})$ carries a canonical atomic equivalence relation $\sim\, \subseteq \underline{A} \times \underline{A}$, for which atomic sheaf semantics validates the transfer and invariance principles.

If $\mathbb{C}$ is a category of epimorphisms with pairings and with **independent pullback** structure then, for all sheaves $\underline{A}, \underline{B}, \underline{C}$ with supports, there is a canonical atomic conditional equivalence relation $\perp\!\!\!\perp_{\underline{A},\underline{B}|\underline{C}} \subseteq \underline{A} \times \underline{B} \times \underline{C}$.

Atomic sheaf semantics validates the axioms for conditional independence including the independence principle.

Another model is the topos of probability sheaves in which $\sim$ is equality-in-distribution and $\perp\!\!\!\perp$ is the usual probabilistic conditional independence relation.

Also the Schanuel topos (equivalent to nominal sets) is a model.