Category Theory 2022-23

Lecture 1

7th October 2022

A *category* $\mathcal{C}$ is given by:

- A collection $|\mathcal{C}|$ of *objects*.
- For every pair X, Y of objects, a collection $\mathcal{C}(X,Y)$ of *morphisms* (*maps*)
- For every object X an *identity morphism* $1_X \in \mathcal{C}(X,X)$
- For every triple X, Y, Z of objects a *composition function* $(-)\circ(-) : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$

Such that:

this means $f \in \mathcal{C}(X,Y)$

- For every X, Y and $X \xrightarrow{f} Y$, $f \circ 1_X = f = 1_Y \circ f$ (identity laws)
- For every $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, $h\circ(g \circ f) = (h \circ g)\circ f$ (associativity)

Exercise The identities are determined uniquely by the composition function.

By the associativity law, chains of morphisms such as

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

yield a unique composite map $X \xrightarrow{h\circ g\circ f} W$.

Equalities between such composites can be expressed as *commutative diagrams*, e.g.,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1_X"'] \arrow[dr, "f"] & Y \arrow[d, "1_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

expresses the identity laws.

Examples

Set The category of sets (and functions).

Objects: sets

Morphisms: $\mathbf{Set}(X,Y)$ = the set of all functions from X to Y (which makes sense as X, Y are sets).

Identities: 1_X = the identity function $x \mapsto x : X \to X$

Composition: $g \circ f$ = the composite function $x \mapsto g(f(x))$

In the category **Set** the collection of objects $|\mathbf{Set}|$ is the collection of all sets, which is not itself a set, rather a proper class.

However, for every X, Y, $\mathbf{Set}(X,Y)$ is a set.

A category $\mathcal{C}$ is locally small if, for every $X, Y \in |\mathcal{C}|$, the "hom set" $\mathcal{C}(X,Y)$ is a set.

$\mathcal{C}$ is small if it is locally small and $|\mathcal{C}|$ is also a set.

The category **Set** is thus locally small but not small.

$\underline{Grp}$ The category of groups (and homomorphisms)

Objects : groups

Morphisms : $Grp(G,H)$ = set of homomorphisms from G to H

Identities and composition : identity functions and function composition (as in $\underline{Set}$)

$\underline{Top}$ The category of topological spaces (and continuous functions)

Objects : topological spaces

Morphisms : $Top(S,T)$ = set of continuous functions from S to T

Identities and composition : function identities & composition

a field

$\underline{Vect}_K$ The category of vector spaces (and linear tranformations) over K

Objects : Vector spaces over K

Morphisms : $Vect_K(U,V)$ = set of linear transformations from U to V

Identities and composition : Function identities & composition.

Commonalities

All examples so far are locally small but not small.

In all, morphisms are classes of functions, with identities and composition given by function identities and composition

The following examples are of a different character

Rel The category of (sets and) relations.

Objects: sets

Morphisms: $\mathbf{Rel}(X,Y) := $ set of relations between X and Y

(Recall a relation between X and Y is a function $R : X \times Y \to \{\text{true}, \text{false}\}$.)

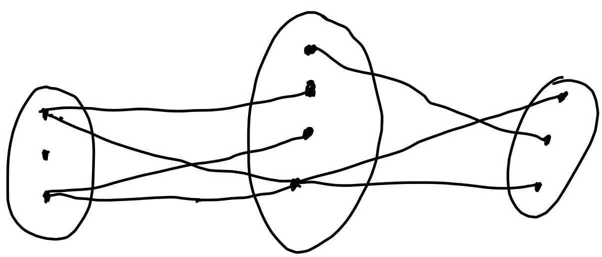
Identities: $1_X = $ the identity relation $x \, 1_X \, x' \Leftrightarrow x = x'$

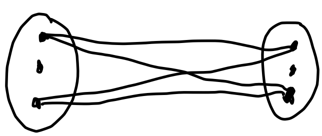
Composition: $S \circ R = $ relation composition $R ; S$

$$x (R;S) z \Leftrightarrow \exists y \; xRy \text{ and } ySz$$

This is again a locally small but not small category. However composition is not function composition.

E.g. A composition $3 \to 4 \to 3$



$=$ 

Let $(G, \cdot, e)$ be any group.

$\underline{G}$ The group G as a category

<u>Objects</u>: Just one object, $*$

<u>Morphisms</u>: $\underline{G}(*, *) := G$

<u>Identities</u>: $1_* := e$

<u>Composition</u>: $y \circ x := y \cdot x$

More generally, the same construction gives a category $\underline{M}$ for any monoid $(M, \cdot, e)$.

$\underline{G}$ (and $\underline{M}$) is a small category.

It has only one object,

Categorification:

Monoid := one object category

Let $(P, \leq)$ be any poset (partially ordered set)

$\underline{P}$ The poset P as a category

Objects: $|\underline{P}| := P$

Morphisms: $\underline{P}(x,y) := \begin{cases} \{*_{x,y}\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

a singleton set

Identities: $1_x := *_{x,x}$

Composition: $*_{y,z} \circ *_{x,y} := *_{x,z}$

More generally, the same construction gives a category for any preorder $(P, \leq)$.

$\underline{P}$ is a small category.

Moreover, there is at most one morphism between any two objects.

Categorification:

preorder := category in which hom sets have at most one element

Special morphisms in a category $\mathcal{C}$

the inverse of f

$X \xrightarrow{f} Y$ is an <u>isomorphism</u> (<u>iso</u>) if there exists $\boxed{Y \xrightarrow{f^{-1}} X}$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$.

Trivially every identity is an isomorphism. It is its own inverse.

<u>Exercises</u>
- If f is an isomorphism then f^{-1} is determined uniquely by f.
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two isomorphisms then $g \circ f$ is also an iso.

In <u>Set</u> the isomorphisms are the bijections

<u>Grp</u>	bijective homomorphisms
<u>Top</u>	homeomorphisms
<u>Vect</u>$_{\mathbb{K}}$	linear isomorphisms
<u>Rel</u>	graphs of bijections

<u>G</u> every morphism is an isomorphism

<u>P</u> (a poset) the only isomorphisms are the identities

Categorification:

group := 1-object category in which every morphism is an isomorphism

poset := category in which hom-sets have at most one morphism and the only isomorphisms are identities.

$X \xrightarrow{f} Y$ is a Monomorphism (mono) if, for every parallel pair $Z \overset{x}{\underset{y}{\rightrightarrows}} X$, $f \circ x = f \circ y \Rightarrow x = y$

Exercises
- Every isomorphism is a monomorphism
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two monos then $g \circ f$ is also mono.
- If $g \circ f$ is mono then so is f.
- If $g \circ f$ is iso and g is mono then f and g are both isos.

In Set the monomorphisms are the injections

Top	continuous injections
Grp	injective homomorphisms
Vect$_K$	injective linear transformations

$\underline{P}$ (a poset) every morphism is a monomorphism

Weekly puzzle In Rel:

- Is there a mono from 2 (a 2-element set) to 1 (a singleton)?
- Is there a mono from 3 to 2?
- Characterise the monomorphisms in Rel.

Solution to Week 1 puzzle :

Characterise Monos in <u>Rel</u>

Consider any map $X \xrightarrow{R} Y$ in <u>Rel</u>

That is, R is a relation given by a function

$$R : X \times Y \longrightarrow 2 \qquad (2 := \{true, false\})$$

Define a function $\tilde{R} : \wp X \to \wp Y$

$$\tilde{R} : X' \longmapsto \{ y \in Y \mid \exists x \in X', \; x R y \}$$

The desired chacterisation is

R is a mono in <u>Rel</u>
$\iff$ the function $\tilde{R}$ is injective

The answers to the other questions are then

There is no mono in <u>Rel</u> from 2 to 1
" " " " " " 3 to 2

(for cardinality reasons there are no injective functions from $\wp 2$ to $\wp 1$ and from $\wp 3$ to $\wp 2$)

Category Theory 2022-23

Lecture 2

14th October 2022

A key philosophy behind category theory is that it is helpful to consider mathematical structures (objects) in combination with a notion of morphism between them.

Categories themselves are a form of mathematical structure.

So what are the morphisms between categories?

A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ ($\mathcal{C}, \mathcal{D}$ categories) is given by:

- a function $F_0 : |\mathcal{C}| \to |\mathcal{D}|$
- for every $X, Y \in |\mathcal{C}|$ a function $F_1 : \mathcal{C}(X,Y) \to \mathcal{D}(F_0 X, F_0 Y)$

such that

(The blue annotations are normally omitted)

- $F_1(1_X) = 1_{F_0 X} \quad \forall X \in |\mathcal{C}|$
- $F_1(g \circ_{\mathcal{C}} f) = (F_1 g) \circ_{\mathcal{D}} (F_1 f) \quad \forall X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$

There is an obvious identity functor from any category to itself. There is also an obvious composite functor $\mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$ for any two functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$. These satisfy the identity and associativity laws.

Exercise Work out the details of the above.

Thus we can form a category whose objects are categories and whose morphisms are functors.

But there are set-theoretic size issues. Categories are in general large structures. Does it make sense to consider a category of all categories and if so is it an object of itself?

We avoid such issues by circumventing them. Define

Cat The category with small categories as objects and functors as morphisms. Forward pointer: Cat should really be defined as a 2-category

This gives us a locally small category of categories.

In spite of the restriction to small categories in Cat it still makes sense to consider functors between arbitrary categories.

Example functors

$U : \underline{Grp} \to \underline{Set}$ the forgetful functor

$U(G, m, e) := G$ object action

$U\left((G,m,e) \xrightarrow{h} (G',m',e')\right) := G \xrightarrow{h} G'$

morphism action

$U : \underline{Top} \to \underline{Set}$

$U : \underline{Vect_k} \to \underline{Set}$

} forgetful functors analogous to the above

If G, H are groups then

functors from $\underline{G}$ to $\underline{H}$ $\cong$ homomorphisms from G to H

(and ditto for monoids)

If P, P' are preorders / posets then

functors from $\underline{P}$ to $\underline{P'}$ $\cong$ monotone (i.e. order-preserving) functions from P to P'.

We have so far seen 2 main kinds of examples of categories

1 - Categories whose objects are mathematical structures, and whose morphisms are transformations / relations between structures

E.g., **Set**, **Grp**, **Top**, $\mathbf{Vect}_K$, **Rel**, **Cat**

Such a category is a single mathematical metastructure that encompasses a whole area of mathematics via its structures (objects) and transformations (morphisms)

Category theory is the "mathematics of mathematics." [E. Cheng]

Watch the video: "What is category theory?"

2 - Individual mathematical structures recast as categories.

E.g. Monoids $\underline{M}$, groups $\underline{G}$

posets/preorders $\underline{P}$

The notion of category axiomatises a very general kind of mathematical structure, of which many familiar mathematical structures arise as natural special cases.

Categorification: derive (standard) mathematics as instances of (sometimes more general) category-theoretic mathematics.

We now start to explore a third rich source of categories

3 — Categories obtained from other categories by category-theoretic constructions

A major part of the power of category theory is that it provides a powerful toolbox of constructions on categories

Opposite (or dual) categories

If $\mathcal{C}$ is a category its *opposite* $\mathcal{C}^{op}$ is defined by:

$$|\mathcal{C}^{op}| := |\mathcal{C}|$$

$$\mathcal{C}^{op}(X,Y) := \mathcal{C}(Y,X)$$

$$1_X \text{ in } \mathcal{C}^{op} := 1_X \text{ in } \mathcal{C}$$

Idea: turn the morphisms around

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}^{op}$ (i.e. $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathcal{C}$)

$$g \circ f \text{ in } \mathcal{C}^{op} := f \circ g \text{ in } \mathcal{C}$$

Examples

For a group G, $(\underline{G})^{op} = \underline{(G^{op})}$ where G^{op} is the *opposite group*.

If G is abelian then $\underline{G} = \underline{G}^{op}$

For a poset/preorder P, $(\underline{P})^{op} = \underline{(P^{op})}$

where P^{op} is the *dual order*

Thus opposite categories generalise standard constuctions of dual/opposite structures

Observe that $(\mathcal{C}^{op})^{op} = \mathcal{C}$. (Taking opposites is an *involution*.)

Insert → Observe functors $C \to D$ in 1-1 correspondence with functors $C^{op} \to D^{op}$

Contravariant Functors

A functor $F : C^{op} \to D$ is called a **contravariant functor** from C to D.

(Ordinary functors $F : C \to D$ are said to be <u>covariant</u>.)

Example: Dual vector space

If V is a vector space over k, recall its <u>dual space</u>

$$V^* := \{ f : V \to k \mid f \text{ is linear} \}$$

The dual space construction is the object action of a contravariant functor from $\underline{\mathrm{Vect}}_k$ to itself.

$$V \mapsto V^* \qquad \text{Object action}$$

$$\begin{array}{c} V \\ \uparrow \lambda \\ W \end{array} \longmapsto \begin{array}{c} V^* \\ \downarrow \\ W^* \end{array} \quad f \mapsto f \circ \lambda \qquad \text{Morphism action}$$

We write this map in its orientation in $\underline{\mathrm{Vect}}_k$.

Example : The Self-duality of $\underline{\mathrm{Rel}}$

If $R : X \times Y \to 2$ is a relation from X to Y

then its <u>transpose</u> $R^T : Y \times X \to 2$ is defined by

$$y R^T x \iff x R y$$

Using this we define a contravariant functor from $\underline{\mathrm{Rel}}$ to itself $T : \underline{\mathrm{Rel}}^{op} \to \underline{\mathrm{Rel}}$

$$X \longmapsto X$$

Object action

$$\begin{tikzcd} X \\ Y \arrow[u, "R"] \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[d, "R^T"'] \\ Y \end{tikzcd}$$

Morphism action

orientation in $\underline{\mathrm{Rel}}$

The same definition gives a functor $T^{op} : \underline{\mathrm{Rel}} \to \underline{\mathrm{Rel}}^{op}$
(a contravariant functor from $\underline{\mathrm{Rel}}^{op}$ to itself)

Notice that $T \circ T^{op} = 1_{\underline{\mathrm{Rel}}}$ and $T^{op} \circ T = 1_{\underline{\mathrm{Rel}}^{op}}$. So $\underline{\mathrm{Rel}} \cong \underline{\mathrm{Rel}}^{op}$.

Exercise (Powerset functors)

1. Find a contravariant functor

$$\underline{\mathrm{Set}}^{op} \longrightarrow \underline{\mathrm{Set}}$$

the powerset of X

whose object action is $X \mapsto \wp X$

2. Find a covariant functor

$$\underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Set}}$$

whose object action is $X \mapsto \wp X$

3. Find a second (i.e., different) solution to question 2.

Product categories

The product $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$ is defined by

$$|\mathcal{C} \times \mathcal{D}| := |\mathcal{C}| \times |\mathcal{D}|$$

$$(\mathcal{C} \times \mathcal{D})((X,Y),(X',Y')) := \mathcal{C}(X,X') \times \mathcal{D}(Y,Y').$$

$$1_{(X,Y)} := (1_X, 1_Y)$$

$$(f,g) \circ (f',g') := (f \circ f', g \circ g')$$

There are evident projection functors

$$\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$$

$$\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

Exercises – Fill in the details

– Generalise to arbitrary finite products $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ and general (indexed) product categories $\prod_{i \in I} \mathcal{C}_i$

It is now clear what is meant by a multi-argument functor

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$$

The hom functor

(This plays a fundamental role in category theory)

If $\mathcal{C}$ is a locally small category then the <u>hom functor</u>

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Set}}$$

is defined by

$$\mathcal{C}(X,Y) := \text{the hom set } \mathcal{C}(X,Y) \qquad \text{object action}$$

$$\begin{array}{ccc} X & & Y \\ g\big\uparrow & & \big\downarrow h \\ X' & & Y' \end{array} \quad \longmapsto \quad \begin{array}{c} \mathcal{C}(X,Y) \\ \big\downarrow \\ \mathcal{C}(X',Y') \end{array} \quad f \longmapsto h \circ f \circ g \qquad \text{Morphism action}$$

<u>Exercise</u> Verify that this is a functor.

The hom functor is contravariant in its first argument and covariant in its second.

Coda: Exploiting duality

Every category-theoretic concept gives rise to a dual concept obtained by interpreting the original concept in the opposite category

E.g. epimorphism (epi) the dual of monomorphism

Quick definition $X \xrightarrow{f} Y$ is an epimorphism in C if f is a monomorphism in C^{op}.

Expanded definition $X \xrightarrow{f} Y$ is an epimorphism if, for all $Y \overset{g}{\underset{h}{\rightrightarrows}} Z$, $\quad g \circ f = h \circ f \Rightarrow g = h$.

From the quick definition, for any property of monomorphisms there is a corresponding dual property of epimorphisms

In Set the epimorphisms are the surjections.

A functor $F: C \to D$ is said to preserve monomorphisms (respectively epimorphisms) if, for every mono (resp. epi) f, it holds that Ff is also mono (resp. epi)

$\boxed{\text{Week 2 puzzle}}$

- Does every functor $F: \underline{Set} \to \underline{Set}$ preserve monos?
- " " " " epis?

Week 2 puzzle solution.

- Does every functor from **Set** to **Set** preserve monos?

Answer: no.

Let A be any fixed set. Then the following action on objects

$$F_A = X \longmapsto \begin{cases} A & \text{if } X = \emptyset \\ 1 & \text{if } X \text{ nonempty} \end{cases}$$

(where 1 is a chosen singleton set) extends to a functor in a unique way. If $|A| \geq 2$ then the functor F_A does not preserve monos because $F_A(\emptyset \to 1) = A \to 1$, where $\emptyset \to 1$ is mono but $A \to 1$ is not.

- Does every functor from **Set** to **Set** preserve epis?

Answer: yes (if the Axiom of choice is assumed).

AC is equivalent to every epi splits. It is easy to verify that functors preserve split epis. (If r has section s then Fr has section Fs.) [I don't know the answer if AC is not assumed!]

Category Theory 2022-23

Lecture 3

21st October 2022

Suppose V is a vector space (over K) of dimension n.

Then V^* is also a vector space of dimension n.

If $e_1, \dots, e_n$ is a basis for V

Then $e_1^*, \dots, e_n^*$ is a basis for V^*

where $e_i^* : (\lambda_1 e_1 + \dots + \lambda_n e_n) \mapsto \lambda_i$

The function

$$(\lambda_1 e_1 + \dots + \lambda_n e_n) \mapsto (\lambda_1 e_1^* + \dots + \lambda_n e_n^*)$$

is then a linear isomorphism from V to V^*

The definition of the isomorphism depends crucially on the initial choice of basis. A different choice of basis results in a different isomorphism.

V^{**} is again a vector space of dimension n

The function

$$v \mapsto \left(f \in V^* \mapsto f(v)\right) \ : \ V \to V^{**}$$

is a linear isomorphism that is defined independtly of any choice of basis.

In avoiding arbitrary choices, the isomorphism $V \to V^{**}$ is <u>natural</u> in a sense that the isomorphism $V \to V^*$ is not.

The isomorphism $V \to V^{**}$ is defined <u>uniformly</u> in V.

Category theory gives a precise interpretation to this idea of naturality / uniformity: the notion of <u>natural transformation</u>.

Natural transformations are morphisms between functors (between the same two categories).

Given functors $F, G : \mathcal{C} \to \mathcal{D}$

a <u>natural transformation</u> $\alpha : F \Rightarrow G$ (or $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$)

the components of α

is a family $\left(FX \xrightarrow{\alpha_X} GX\right)_{X \in |\mathcal{C}|}$ of

morphisms in $\mathcal{D}$ indexed by objects of $\mathcal{C}$

satisfying :

for every morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$,

the square in $\mathcal{D}$ below commutes

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

(i.e., $\alpha_Y \circ Ff = Gf \circ \alpha_X$)

the <u>naturality</u> <u>condition</u>

For our vector space example, define

$$\varepsilon_V : v \longmapsto (f \in V^* \mapsto f(v))$$

which gives a family $(V \xrightarrow{\alpha_V} V^{**})_{V \in |\underline{\mathrm{Vect}}|}$ indexed by vector spaces (no need to require a finite dimension).

We show that this defines a natural transformation

$$\begin{tikzcd} \underline{\mathrm{Vect}} \arrow[rr, bend left, "1_{\underline{\mathrm{Vect}}}"] \arrow[rr, bend right, "(\cdot)^{**}"'] \arrow[rr, phantom, "\Downarrow \varepsilon"] & & \underline{\mathrm{Vect}} \end{tikzcd}$$

where $(\cdot)^{**}$ is the composite functor $\underline{\mathrm{Vect}} \xrightarrow{(\cdot)^*} \underline{\mathrm{Vect}}^{op} \xrightarrow{(\cdot)^*} \underline{\mathrm{Vect}}$

defined last week

The resulting action of $(\cdot)^{**}$ on morphisms is

$$\begin{tikzcd} V \arrow[d, "\ell"'] & \longmapsto & V^{**} \arrow[d, "\ell^{**} := F \in V^{**} \mapsto (g \in W^* \mapsto F(g \circ \ell))"] \\ W & & W^{**} \end{tikzcd}$$

Verifying naturality

For any $V \xrightarrow{l} W$ in <u>Vect</u>, we need to show that the square below commutes

$$\begin{tikzcd}
V \arrow[r, "\varepsilon_V"] \arrow[d, "l"'] & V^{**} \arrow[d, "l^{**}"] \\
W \arrow[r, "\varepsilon_W"'] & W^{**}
\end{tikzcd}$$

Indeed, for any $v \in V$, we have.

$$\begin{aligned}
(l^{**} \circ \varepsilon_V)(v) &= l^{**}(f \mapsto f(v)) \\
&= g \mapsto (f \mapsto f(v))(g \circ l) \\
&= g \mapsto g(l(v)) \\
&= (\varepsilon_W \circ l)(v)
\end{aligned}$$

For any set X, consider the functions

$$\{\cdot\}_X : x \mapsto \{x\} : X \to \mathcal{P}X$$

union of a set of subsets of X

$$\bigcup_X : \underline{Y} \mapsto \bigcup \underline{Y} : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$$

These are defined in a uniform way for all sets X!

The maps give the components of natural transformations

$$\{\cdot\} : 1_{\underline{\text{set}}} \Rightarrow \mathcal{P}$$

$$\bigcup : \mathcal{P}^2 \Rightarrow \mathcal{P} \qquad (\mathcal{P}^2 := \mathcal{P}\mathcal{P})$$

where $\mathcal{P}$ is the <u>covariant powerset functor</u>

$$X \mapsto \mathcal{P}X$$

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{P}X \\ \downarrow \mathcal{P}_f := X' \mapsto f(X') \\ \mathcal{P}Y \end{array} \qquad (f(X') := \{f(x) \mid x \in X'\} \text{ the direct image})$$

(This answers part of an exercise from last week.)

Verifying naturality

For any $X \xrightarrow{f} Y$ in $\underline{Set}$, one needs to check the commutativity of the squares below.

$$\begin{tikzcd}
X \arrow[r, "\{\cdot\}"] \arrow[d, "f"'] & \mathcal{P}X \arrow[d, "x' \mapsto f(x')"] \\
Y \arrow[r, "\{\cdot\}"'] & \mathcal{P}Y
\end{tikzcd}$$

$\mathcal{P}f$

$$\begin{tikzcd}
\mathcal{P}^2X \arrow[r, "\cup"] \arrow[d, "\underline{Y} \mapsto \{f(x') \mid x' \in \underline{Y}\}"'] & \mathcal{P}X \arrow[d, "x' \mapsto f(x')"] \\
\mathcal{P}^2Y \arrow[r, "\cup"'] & \mathcal{P}Y
\end{tikzcd}$$

$\mathcal{P}^2 f$

Given

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4, "\alpha\Downarrow"] \arrow[r, phantom, shift right=4, "\beta\Downarrow"] & \mathcal{D}
\end{tikzcd}$$

Vertical composition of natural transformations

There is a composite nat. trans.

$$\beta \circ \alpha : F \Rightarrow H$$

defined by

composition in $\mathcal{D}$.

$$(\beta \circ \alpha)_X := \beta_X \circ \alpha_X$$

Exercise Verify the naturality condition.

Functor categories

Given categories $\mathcal{C}, \mathcal{D}$, the <u>functor category</u> $[\mathcal{C}, \mathcal{D}]$ is defined by:

$$|[\mathcal{C}, \mathcal{D}]| := \text{functors } \mathcal{C} \to \mathcal{D}$$

$$[\mathcal{C}, \mathcal{D}](F, G) := \text{nat transformations } F \Rightarrow G$$

$$1_F := \text{identity transformation } \{FX \xrightarrow{1_{FX}} FX\}_{X \in |\mathcal{C}|}$$

$$\beta \circ \alpha := \text{vertical composition}$$

There are size issues! If $\mathcal{C}, \mathcal{D}$ are large then $[\mathcal{C}, \mathcal{D}]$ is very large. However,

- $\mathbb{C}, \mathbb{D}$ small $\Rightarrow$ $[\mathbb{C}, \mathbb{D}]$ small
- $\mathbb{C}$ small, $\mathcal{D}$ locally small $\Rightarrow$ $[\mathbb{C}, \mathcal{D}]$ locally small.

Whiskering

Given
$$\begin{tikzcd} B \arrow[r, "F"] & C \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\alpha\Downarrow"] & D \arrow[r, "H"] & F \end{tikzcd}$$

Define $H\alpha : HG_1 \Rightarrow HG_2$ by $(H\alpha)_X := H\alpha_X$

$\alpha F : G_1F \Rightarrow G_2F$ by $(\alpha F)_X := \alpha_{FX}$

Either way of combing leads to the same

$$H\alpha F : HG_1F \Rightarrow HG_2F$$

Horizontal composition

Given
$$\begin{tikzcd} C \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd}$$

Define $\beta * \alpha : G_1F_1 \Rightarrow G_2F_2$

by $\beta * \alpha := (\beta F_2) \circ (G_1\alpha)$ $\left(= (G_2\alpha) \circ (\beta F_1) \right)$

Exercise Using the above show that functor composition is itself a functor $G, F \mapsto GF : [D, E] \times [C, D] \to [C, E]$.

Proposition

The following are equivalent for a natural transformation
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

(1) Every component $FX \xrightarrow{\alpha_X} GX$ is an isomorphism in $\mathcal{D}$

(2) α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$.

To prove (1) $\Rightarrow$ (2), the main point is that the inverses $\left(FX \xleftarrow{\alpha_X^{-1}} GX\right)_{X \in |\mathcal{C}|}$ satisfy the naturality condition.

Exercise!

A natural transformation enjoying property (1) (or (2)) is called a <u>natural isomorphism</u>

Our vector space example

$$\left(V \xrightarrow{\varepsilon_V} V^{**} \right)_{V \in \underline{\text{FDVect}}}$$

the category of finite dimensional vector spaces

is a natural isomorphism when restricted to finite dimensional vector spaces.

The inclusion functor

$$\begin{tikzcd} \underline{\text{FDVect}} \arrow[rr, bend left, "I"] \arrow[rr, bend right, "(\cdot)^{**}"'] \arrow[rr, phantom, "\Downarrow \varepsilon"] & & \underline{\text{Vect}} \end{tikzcd}$$

or $\underline{\text{FDVect}}$

Further constructions of categories.

Let $\mathcal{C}$ be a category and $I \in |\mathcal{C}|$.

<u>The slice category</u> (of $\mathcal{C}$ over I) $\mathcal{C}/I$

$$|\mathcal{C}/I| := \text{maps in } \mathcal{C} \text{ with codomain } I$$

$$\mathcal{C}/I\left(\begin{array}{c} X \\ {\scriptstyle p}\downarrow \\ I \end{array}, \begin{array}{c} Y \\ {\scriptstyle q}\downarrow \\ I \end{array}\right) := \left\{ X \xrightarrow{f} Y \;\middle|\; \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I & \end{tikzcd} \text{ commutes} \right\}$$

Identities and composition inherited from $\mathcal{C}$

<u>The coslice category</u> (of $\mathcal{C}$ under I) $I/\mathcal{C}$

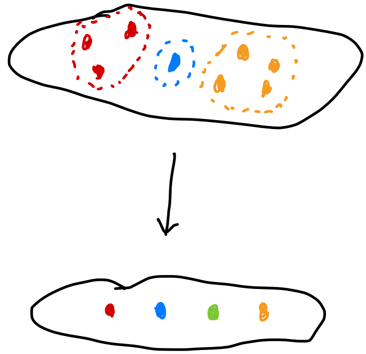
$$|I/\mathcal{C}| := \text{maps in } \mathcal{C} \text{ with domain } I$$

$$I/\mathcal{C}\left(\begin{array}{c} I \\ {\scriptstyle i}\downarrow \\ X \end{array}, \begin{array}{c} I \\ \downarrow{\scriptstyle j} \\ Y \end{array}\right) := \left\{ X \xrightarrow{f} Y \;\middle|\; \begin{tikzcd} & I \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} \text{ commutes} \right\}$$

Identities and composition inherited from $\mathcal{C}$.

There are forgetful functors $\mathcal{C}/I \to \mathcal{C}$ and $I/\mathcal{C} \to \mathcal{C}$

A function $\begin{array}{c} X \\ \downarrow{\scriptstyle p} \\ I \end{array}$



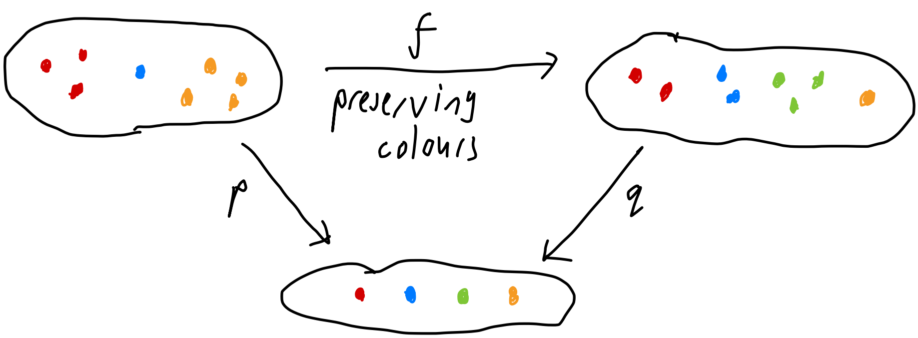
Gives rise to a function

$$i \mapsto p^{-1}(i) : I \to \mathcal{P}X$$

mapping every $i \in I$ to its fibre

Maps in $\underline{\mathrm{Set}}/I$ from $\begin{array}{c} X \\ {\scriptstyle p}\downarrow \\ I \end{array}$ to $\begin{array}{c} Y \\ \downarrow{\scriptstyle q} \\ I \end{array}$

are functions $X \xrightarrow{f} Y$ that map each $p^{-1}(i)$ to $q^{-1}(i)$



$\equiv$ly families of functions $(f_i : p^{-1}(i) \to q^{-1}(i))_{i \in I}$

The category $\underline{\mathrm{Fam}}_I$ of I-indexed families of sets

<u>Objects</u> $|\underline{\mathrm{Fam}}_I| :=$ the collection of I-indexed families of sets $(X_i)_{i \in I}$

$= I \to \underline{\mathrm{Set}}$

<u>Morphisms</u>

$$\underline{\mathrm{Fam}}_I\big((X_i)_{i\in I}, (Y_i)_{i \in I}\big) :=$$

I-indexed families of functions

$$(f_i : X_i \to Y_i)_{i \in I}$$

$$= \prod_{i \in I} X_i \to Y_i$$

<u>Identities</u> $1_{(X_i)_{i \in I}} := (1_{X_i} : X_i \to X_i)_{i \in I}$

<u>Composition</u> $(g_i)_{i\in I} \circ (f_i)_{i \in I} := (g_i \circ f_i)_{i \in I}$.

Shorter description:

$$\underline{\mathrm{Fam}}_I := [\underline{I}, \underline{\mathrm{Set}}]$$

The <u>discrete</u> category

$|\underline{I}| := I$

only identity morphisms

A functor $F : \underline{\text{Set}}/I \to \underline{\text{Fam}}_I$

$$\begin{array}{c} X \\ p\downarrow \\ I \end{array} \longmapsto \left(p^{-1}(i)\right)_{i\in I} \qquad \text{object action}$$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I & \end{tikzcd} \mapsto \left(f\restriction_{p^{-1}(i)} : p^{-1}(i) \to q^{-1}(i)\right)_{i\in I} \qquad \text{morphism action}$$

A functor $S : \underline{\text{Fam}}_I \to \underline{\text{Set}}/I$

$$(X_i)_{i\in I} \longmapsto \begin{array}{c} \sum_{i\in I} X_i \\ \downarrow \pi_1 \\ I \end{array} \begin{array}{l} := \{(i,x) \mid i\in I, x\in X_i\} \\ := (i,x) \mapsto i \end{array} \qquad \text{object action}$$

$$(f_i)_{i\in I} \mapsto \left((i,x) \mapsto (i, f_i(x))\right)_{i\in I} \qquad \text{morphism action}$$

A natural isomorphism $\phi : 1_{\underline{\text{Set}}/I} \Rightarrow SF$

$$\phi_{\begin{array}{c} X \\ \downarrow p \\ I \end{array}} := x\in X \longmapsto (p(x), x) : \begin{array}{c} X \\ p\downarrow \\ I \end{array} \to \begin{array}{c} \sum_{i\in I} p^{-1}(x) \\ \downarrow \pi_1 \\ I \end{array}$$

A natural isomorphism $\psi : 1_{\underline{\text{Fam}}_I} \Rightarrow FS$

$$\psi_{(X_i)_{i\in I}} := \left(x\in X_i \longmapsto (i,x)\right)_{i\in I} : (X_i)_{i\in I} \to (\{i\}\times X_i)_{i\in I}$$

An equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$ is given by:

- Functors $I : \mathcal{C} \to \mathcal{D}$ and $J : \mathcal{D} \to \mathcal{C}$
- Natural isomorphisms $\alpha : 1_{\mathcal{C}} \Rightarrow JI$ and $\beta : 1_{\mathcal{D}} \Rightarrow IJ$

So the categories $\underline{\mathrm{Set}}/I$ and $\underline{\mathrm{Fam}}_I$ are equivalent.

Week 3 puzzle

Find descriptions of the following functor categories in more direct (and perhaps familiar) mathematical terms.

(1) $[\underline{G}, \underline{\text{Set}}]$ for any group G.

(2) $[\underline{\Gamma}, \underline{\text{Set}}]$

where $\underline{\Gamma}$ is the 2-object category below with only 2 non-identity morphisms

$$E \underset{t}{\overset{s}{\rightrightarrows}} V$$

Week 3 puzzle solution

The functor category $[\underline{G}, \underline{Set}]$ ($(G, e, \cdot)$ a group)

A functor $F : \underline{G} \to \underline{Set}$ consists of

- A set X (the set to which the unique object $* \in |\underline{G}|$ is mapped by F)

- Functions $F_g : X \to X$ for every $g \in G$ satisfying $F_e = 1_X$ and $F_{(g \cdot h)} = F_g \circ F_h$

If we write $F_g(x)$ as $g \bullet x$ then we see that a functor corresponds to $(X, \bullet)$ where X is a set and $\bullet : G \times X \to X$ satisfies

$$e \bullet x = x$$
$$(g \cdot h) \bullet x = g \bullet (h \bullet x)$$

I.e., a functor $[\underline{G}, \underline{Set}]$ is a <u>left G action</u>.

A natural transformation $\alpha : F \Rightarrow F' : \underline{G} \to \underline{Set}$

Is given by a function $F* \xrightarrow{f} F'*$

satisfying the naturality property, for any

$g \in G$

$$\begin{tikzcd} F* \arrow[r, "f"] \arrow[d, "Fg"'] & F'* \arrow[d, "F'g"] \\ F* \arrow[r, "f"'] & F'* \end{tikzcd}$$

With functors viewed as left actions, a natural transformation from $(X, \bullet)$ to $(X', \bullet')$ is a function $f : X \to X'$ s.t.

$$\forall x \in X \quad \forall g \in G \quad f(g \bullet x) = g \bullet' f(x)$$

i.e. an <u>equivariant function</u>

So $[\underline{G}, \underline{Set}]$ is the category of left G-actions (objects) and equivariant functions (morphisms)

The functor category $[\underline{\Delta}, \underline{Set}]$ $\left(\underline{\Delta} = e \overset{s}{\underset{t}{\rightrightarrows}} v\right)$

A functor $F : \underline{\Delta} \to \underline{Set}$ consists of

- Two sets E and V (given as $F(e)$ and $F(v)$)
- Two functions $\sigma, \tau : E \to V$ (given as $F(s)$, $F(t)$).

Such a structure defines a graph (quiver).
E is the set of directed edges, V the set of vertices, and σ & τ respectively map an edge to its source & target.

A natural transformation $\alpha : F \Rightarrow F'$
where F is (E, V, σ, τ) and F' is (E', V', σ', τ')
is given by a pair of functions $g : E \to E'$ and $f : V \to V'$
such that

$$\sigma'(g(e)) = f(\sigma(e))$$
$$\tau'(g(e)) = f(\tau(e))$$

i.e. a natural transformation is a graph homomorphism.

$[\underline{\Delta}, \underline{Set}]$ is the category of graphs (quivers) and graph homomorphisms.

Strictly it is equivalent to $\underline{Grph}$ as defined in Lecture 5, due to the inessential difference between $E \overset{\sigma}{\underset{\tau}{\rightrightarrows}} V$ and $(G(u,v))_{u,v \in |V|}$ (cf. $\underline{Set}/I$ vs. $\underline{Fam}_I$)

Category Theory 2022-23

Lecture 4

28^{th} October 2022

$\underline{\mathrm{Mat}_K}$ - the category of (natural numbers and) matrices over K

$$|\mathrm{Mat}_K| := \mathbb{N}$$

$$\begin{array}{ccc} & \text{rows} & \text{columns} \end{array}$$

$\underline{\mathrm{Mat}_K}(m,n)$:= the set of $n \times m$ matrices with entries from K

Identities $1_n := I_n$ ($n \times n$ identity matrix)

Composition $B \circ A := BA$ matrix multiplication

A functor $J : \mathrm{Mat}_K \to \mathrm{Vect}_K$

$$n \mapsto K^n$$

$$\begin{array}{ccc} m & & K^m \\ A \big\downarrow & \mapsto & \big\downarrow JA := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \mapsto A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ n & & K^n \end{array}$$

The functor $J : \underline{\mathrm{Mat}_K} \to \underline{\mathrm{Vect}_K}$ is full and faithful (it is an embedding of categories)

Moreover $J : \underline{\mathrm{Mat}_K} \to \underline{\mathrm{FDVect}_K}$ is also essentially surjective on objects (it is a weak equivalence of categories)

A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- <u>full</u> if, for every $X, Y \in |\mathcal{C}|$, the morphism action $F : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is surjective
- <u>faithful</u> if, for every $X, Y \in |\mathcal{C}|$, the morphism action $F : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is injective

A functor that is full and faithful is sometimes called an <u>embedding</u> of categories.

<u>Lemma</u> If F is an embedding then F reflects isos (i.e. for any $X \xrightarrow{f} Y$ in $\mathcal{C}$, if Ff is iso then so is f).

(Functors that reflect isos are sometimes called <u>conservative</u>.)

<u>Proof</u> A worthwhile exercise.

A functor $F : C \to D$ is:

- <u>essentially surjective on objects</u> if, for every $Y \in |D|$ there exists $X \in C$ s.t. $FX \cong Y$

FX is isomorphic to Y; i.e. there exists an iso $FX \to Y$ in D

A functor that is full, faithful and essentially surjective on objects is called a <u>weak equivalence</u>

<u>Theorem</u>

1. If F is part of an equivalence of categories (F, G, α, β) then F is a weak equivalence.

2. If F is a weak equivalence then assuming a suitable version of the axiom of choice F arises as part of an equivalence of categories (F, G, α, β).

Outline proof of (2)

Using essential surjectivity and choice, choose, for any $Y \in |D|$, an object $GY \in |C|$ with iso $Y \xrightarrow{\beta_Y} FGY$.

For any $Y \xrightarrow{g} Y'$ in D, $h = \beta_{Y'} \circ g \circ \beta_Y^{-1}$ is the unique map $FGY \xrightarrow{h} FGY'$ s.t.

$$\begin{tikzcd} Y \arrow[r, "\beta_Y"] \arrow[d, "g"'] & FGY \arrow[d, "h"] \\ Y' \arrow[r, "\beta_{Y'}"'] & FGY' \end{tikzcd} \qquad (*)$$

commutes.

Since $F : C(GY, GY') \to D(FGY, FGY')$ is a bijection (F an embedding) there is a unique map $GY \xrightarrow{Gg} GY'$ in C such that $h = FGg$. The assignment $g \mapsto Gg$ is functorial and $(*)$ shows that $\beta : 1_D \Rightarrow FG$ is natural.

To define α, consider, for any $X \in |C|$ the function $F : C(X, GFX) \to C(FX, FGFX)$. Since F is an embedding let $X \xrightarrow{\alpha_X} GFX$ be the unique map s.t. $F\alpha_X = \beta_{FX}$. Since F reflects isos, α_X is iso. One then checks that $\alpha : 1_C \Rightarrow GF$ is natural. $\square$

The highlighted claims are left as exercises

What is meant by a suitable version of the axiom of choice?

If $\mathcal{C}, \mathcal{D}$ are locally small categories, we need the <u>axiom of global choice</u>.

If $\mathcal{C}, \mathcal{D}$ are small categories, we need the usual <u>axiom of choice</u>, which has a nice category-theoretic formulation:

(AC) Every epimorphism in <u>Set</u> splits

Suppose $X \xrightarrow{s} Y \xrightarrow{r} X$ in a category $\mathcal{C}$ are such that $r \circ s = 1_X$. Then r is epi and s is mono (exercise)

An epi $Y \xrightarrow{r} X$ is said to <u>split</u> if $\exists\, X \xrightarrow{s} Y$ s.t. $r \circ s = 1_X$
A mono $X \xrightarrow{s} Y$ " <u>split</u> " $\exists\, Y \xrightarrow{r} X$ s.t. $r \circ s = 1_X$

When one has $X \xrightarrow{s} Y \xrightarrow{r} X$ with $r \circ s = 1_X$, then Y is said to be a <u>retract</u> of X. The split epi $Y \xrightarrow{r} X$ is called a <u>retraction</u> and the split mono $X \xrightarrow{s} Y$ is called a <u>section</u>. The composite $t = s \circ r : Y \to Y$ is an <u>idempotent</u>: $t \circ t = t$

Pullbacks

In a category $\mathcal{C}$, a <u>pullback</u> of a <u>cospan</u> $X \xrightarrow{f} Z \xleftarrow{g} Y$ is given by a <u>span</u> $X \xleftarrow{p} P \xrightarrow{q} Y$ for which $f \circ p = g \circ q$ and such that, for any span $X \xleftarrow{x} W \xrightarrow{y} Y$ with $f \circ x = g \circ y$, there exists a unique $W \xrightarrow{w} P$ such that $p \circ w = x$ and $q \circ w = y$.

$$
\begin{tikzcd}
W \arrow[dr, dashed, "w"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

A category $\mathcal{C}$ is said to <u>have pullbacks</u> if every span in $\mathcal{C}$ has a pullback.

<u>Set</u> has pullbacks.

Given $X \xrightarrow{f} Z \xleftarrow{g} Y$, define

$$P := \{(x,y) \in X \times Y \mid f(x) = g(y)\}$$

$$p := (x,y) \mapsto x \quad : P \to X$$

$$q := (x,y) \mapsto y \quad : P \to Y$$

Then

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

notation for pullback square

Intuition 1: Fibred Products

Given $X \xrightarrow{f} I \xleftarrow{g} Y$ in $\underline{Set}$

The pullback P can be defined as a family of products of fibres

$$P = \biguplus_{i \in I} f^{-1}(i) \times g^{-1}(i)$$

with projections

$$\pi_1 : P \to X$$
$$\pi_2 : P \to Y$$

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & I \end{tikzcd}$$

The diagonal map gives P as the I-indexed family $\left(f^{-1}(i) \times g^{-1}(i)\right)_{i \in I}$.

Intuition 2: reindexing

Given $J \xrightarrow{r} I \xleftarrow{f} X$ in <u>Set</u>

The pullback can be defined as

$$P := \sum_{j \in J} f^{-1}(r(j))$$

$$f' := (j, x) \mapsto j \;:\; P \to J$$

$$p := (j, x) \mapsto x \;:\; P \to X$$

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
J \arrow[r, "r"'] & I
\end{tikzcd}$$

The left hand edge gives P as the J-indexed family

$$\left(f^{-1}(r(j))\right)_{j \in J}$$

Intuition 3 : inverse image

Given $X \xrightarrow{f} Y$ and $Y' \subseteq Y$ in **Set**

the following is a pullback

$$\begin{tikzcd}
f^{-1}(Y') \arrow[r, "f\restriction_{f^{-1}(Y')}"] \arrow[d, "i'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "i"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where the vertical maps are the inclusion functions.

(up to isomorphism, this is a special case of intuition 2.)

Intuition 4 : Intersection

Given $X' \subseteq X \supseteq X''$ in $\underline{\text{Set}}$

The following is a pullback

$$\begin{tikzcd}
X' \cap X'' \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, hook] \\
X'' \arrow[r, hook] & X
\end{tikzcd}$$

where all maps are inclusion functions.

(This is, up to isomorphism, a special case of intuition 1.)

Uniqueness up to isomorphism

If
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

are both pullback squares then the unique map $P' \xrightarrow{i} P$ s.t $p \circ i = p'$ and $q \circ i = q'$ is an iso.

Joint Mon(omorph)icity

If
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
is a pullback then, for any $w, v : W \to P$,

$(p \circ w = p \circ v$ and $q \circ w = q \circ v) \Rightarrow w = v$

the maps p, q are jointly mono

Preservation of Monos

If:
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

notation for mono

is a pullback square and f is mono then so is q.

The pullback lemma

Given a commuting diagram in a category $\mathcal{C}$ of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

1. If $\boxed{A}$ and $\boxed{B}$ are both pullbacks then so is $\boxed{AB}$.

2. If $\boxed{AB}$ and $\boxed{B}$ are both pullbacks then so is $\boxed{A}$

We shall prove this next week.
In the meantime, you can try to prove it yourself

Subobjects

In any category $\mathcal{C}$ define a relation $\sqsubseteq$ on the set of monomorphisms into $X \in |\mathcal{C}|$ by

$$Y \overset{m}{\rightarrowtail} X \sqsubseteq Y' \overset{m'}{\rightarrowtail} X \iff$$

$\exists\, l : Y \to Y'$ s.t.
$$\begin{tikzcd} Y \arrow[r, "l"] \arrow[dr, tail, "m"'] & Y' \arrow[d, tail, "m'"] \\ & X \end{tikzcd}$$
commutes

There exists at most one l as above and it is always mono.
The relation $\sqsubseteq$ is a preorder (reflexive and transitive) on monos into X.
Define $m \equiv m' \iff m \sqsubseteq m'$ and $m' \sqsubseteq m$. This is an equivalence relation, and $m \equiv m' \iff$ the unique l as above is iso (Exercise)

A <u>subobject</u> of X is an equivalence class of monos into X under $\equiv$. The collection $\mathrm{Sub}(X)$ of subobjects into X is partially ordered by $\sqsubseteq$.

Week 4 puzzle

(1) For any set X, describe the partial order $\mathrm{Sub}(X)$ (up to isomorphism of partial orders) arising from the category <u>Set</u> in direct (and familiar) mathematical terms.

(2) Ditto for any vector space V w.r.t. the category <u>Vect</u>$_K$

Category Theory 2022-23

Lecture 5

4th November 2022

Recall A <u>pullback</u> of $X \xrightarrow{f} Z \xleftarrow{g} Y$ is given by

- $X \xleftarrow{p} P \xrightarrow{q} Y$ s.t. $f \circ p = g \circ q$, and

satisfying

- for any $X \xleftarrow{x} W \xrightarrow{y} Y$ s.t. $f \circ x = g \circ y$, there exists a unique $W \xrightarrow{w} P$ such that $p \circ w = x$ and $q \circ w = y$.

$$\begin{tikzcd}
W \arrow[dr, dashed, "w"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

The pullback lemma

Given a commuting diagram in a category $\mathcal{C}$ of the form

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & {} \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "B"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

1. If $\downarrow \overset{\rightarrow}{\underset{\rightarrow}{A}} \downarrow$ and $\downarrow \overset{\rightarrow}{\underset{\rightarrow}{B}} \downarrow$ are both pullbacks then so is $\downarrow \overset{\longrightarrow}{\underset{\longrightarrow}{AB}} \downarrow$.

2. If $\downarrow \overset{\longrightarrow}{\underset{\longrightarrow}{AB}} \downarrow$ and $\downarrow \overset{\rightarrow}{\underset{\rightarrow}{B}} \downarrow$ are both pullbacks then so is $\downarrow \overset{\rightarrow}{\underset{\rightarrow}{A}} \downarrow$

(A more general version of 2:

If $\downarrow \overset{\longrightarrow}{\underset{\longrightarrow}{AB}} \downarrow$ is a pullback and p, q are jointly mono then $\downarrow \overset{\rightarrow}{\underset{\rightarrow}{A}} \downarrow$ is a pullback.)

Proof of generalised 2 We have a commuting diagram

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"'] \arrow[dr, phantom, "A"] & X' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "B"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "g'"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

where AB is a pullback and f', h are jointly mono. We need to show that A is a pullback.

Consider any $Y'' \xleftarrow{y''} W \xrightarrow{x'} X'$ s.t. $f'x' = g'y''$. We must show there is a unique $W \xrightarrow{w} X''$ s.t. $f''w = y''$ and $h'w = x'$. Note that any such w also satisfies $hh'w = hx'$.

We have $fhx' = gf'x' = gg'y''$, so since AB is a pullback there exists a unique $W \xrightarrow{w} X''$ s.t. $f''w = y''$ and $hh'w = hx'$. By what we noted above, it is enough to show that this w also satisfies $h'w = x'$.

We now use the property that f' and h are jointly mono. Since $hh'w = hx'$ and $f'h'w = g'f''w = g'y'' = f'x'$

it follows that $h'w = x'$, as required.

$\square$

Binary product

A *binary product* of $X, Y \in |\mathcal{C}|$ is given by a span $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ in $\mathcal{C}$ such that, for every span $X \xleftarrow{x} Z \xrightarrow{y} Y$ in $\mathcal{C}$ there exists a unique map $Z \xrightarrow{z} P$ s.t. $p_1 \circ z = x$ and $p \circ z = y$.

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "x"] \arrow[r, dotted, "z"] \arrow[dr, "y"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & Y
\end{tikzcd}$$

Proposition If $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$ and $X \xleftarrow{p_1'} P' \xrightarrow{p_2'} Y$ are both binary products then the unique map $P' \xrightarrow{i} P$ such that $p_1 \circ i = p_1'$ and $p_2 \circ i = p_2'$ is an isomorphism.

We say that $\mathcal{C}$ *has binary products* if, for every pair $X, Y \in |\mathcal{C}|$, a product $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$ exists in $\mathcal{C}$.

Notation It is common to write $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ for a chosen binary product for X, Y in $\mathcal{C}$, and to write (x, y) for the unique $Z \to X \times Y$ given by $X \xleftarrow{x} Z \xrightarrow{y} Y$ as above.

I-indexed product

The *product* of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ is given by an object $P \in |\mathcal{C}|$ and a family $(P \xrightarrow{p_i} X_i)_{i\in I}$ of maps in $\mathcal{C}$ such that, for every family $(Z \xrightarrow{x_i} X_i)_{i\in I}$, there exists a unique map $Z \xrightarrow{z} P$ s.t $\forall i \in I \quad p_i \circ z = x_i$.

$$\begin{tikzcd} Z \arrow[r, dashed, "z"] \arrow[dr, "x_i"'] & P \arrow[d, "p_i"] \\ & X_i \end{tikzcd} \qquad \forall i$$

I-indexed products are determined up to isomorphism.

- We say $\mathcal{C}$ has (small) products if, for every set I, every I-indexed family of objects $(X_i)_{i\in I}$ has a product.
- We say $\mathcal{C}$ has finite products if, for every finite set I, every I-indexed family of objects $(X_i)_{i\in I}$ has a product.

Notation: Common to write $\left(\prod_{i\in I} X_i \xrightarrow{\pi_i} X_i\right)_{i\in I}$ for a chosen product of $(X_i)_{i\in I}$.

We also write $(f_i)_{i\in I}$ for the unique map $Z \longrightarrow \prod_{i\in I} X_i$ such that $\pi_i \circ (f_i)_{i\in I} = f_i \quad \forall i \in I$

Set has products. $\prod_{i\in I} X_i := \{(x_i)_{i\in I} \mid \forall i\in I \; x_i \in X_i\}$

$$\pi_i := (x_i)_{i\in I} \mapsto x_i : \prod_{i\in I} X_i \to X_i \quad \text{(as expected!)}$$

Vect$_k$ has products $\prod_{i\in I} V_i :=$ the cartesian product of vector spaces

Grp has products $\prod_{i\in I} G_i :=$ the direct product of groups

Top has products $\prod_{i\in I} S_i =$ the topological product of spaces.

Verification that topological products are (categorical) products

Recall the topological product $\prod_{i\in I} S_i$ endows the product set with the coarsest topology that makes every projection $\prod_{i\in I} S_i \xrightarrow{\pi_i} S_i$ continuous.

I.e., the topology on $\prod_{i\in I} S_i$ has sub-basis $\{\pi_i^{-1}(U) \mid i\in I,\ U \text{ an open subset of } S_i\}$

Given a top. space Z and continuous $Z \xrightarrow{f_i} S_i \quad \forall i\in I$

There is a unique function $Z \xrightarrow{g} \prod_{i\in I} S_i$ s.t. $f_i = \pi_i \circ g \; \forall i$

namely $g(z) = (f_i(z))_{i\in I}$.

We need to show g is continuous. This is true because, for any sub-basic open subset $\pi_i^{-1}(U) \subseteq \prod_{i\in I} S_i$, we have

$$g^{-1}(\pi_i^{-1}(U)) = f_i^{-1}(U) \text{ which is open in } Z \text{ because } f_i \text{ is continuous.}$$

Terminal object

A special case of I-indexed product : $I = \emptyset$

There is exactly one empty family of objects.

Its product is an object T such that, for any $Z \in |\mathcal{C}|$ there exists a unique morphism from Z to T.

Terminal objects are determined up to isomorphism

<u>Notation</u> Common to write 1 for a terminal object and $Z \xrightarrow{!_Z} 1$ for the unique map.

<u>Proposition</u> A category has finite products if and only if it has binary products and a terminal object.

<u>Proof</u> $\Rightarrow$ is immediate as binary products and terminal object are special cases of finite products.

$\Leftarrow$ We construct $\prod_{i \in I} (X_i)$ by induction on $n := |I|$.

<u>$n=0$</u> The product is the terminal object

<u>$n=1$</u> A singleton product $\prod_{i=1}^{1} X_i$ is $(X_i \xrightarrow{1_{X_i}} X_i)_{i=1}^{1}$

<u>$n>1$</u> Construct $\prod_{i=1}^{n} X_i$ as $\left(\prod_{i=1}^{n-1} X_i\right) \times X_n$ using the induction hypothesis and a binary product. $\square$

Equalisers

An *equaliser* of a parallel pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in C is a map $E \xrightarrow{e} X$ such that:

- $f \circ e = g \circ e$, and
- for any $Z \xrightarrow{x} X$ such that $f \circ x = g \circ x$, there exists a unique $Z \xrightarrow{z} E$ such that $e \circ z = x$

$$\begin{tikzcd} & E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[ur, dashed, "z"] \arrow[urr, "x"'] & & & \end{tikzcd}$$

Equalizers are determined up to isomorphism.

We say that C *has equalisers* if every parallel pair of maps in C has an equaliser.

Proposition Equaliser maps $E \xrightarrow{e} X$ are always mono.

A morphism in a category is called a *regular mono* if there exists some parallel pair for which it is an equaliser map.

Exercise Split monos are regular.

Set has equalisers

Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ define $E := \{x \in X \mid f(x) = g(x)\}$
$e := x \mapsto x : E \to X$

(**Vect_k** and **Grp** have equalisers.)

Top has equalisers.

Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ continuous functions between topological spaces

define E as above endowed with the <u>subspace topology</u> (from X).

Exercises

- In **Set** every monomorphism is regular.

- A continuous function is a regular mono in **Top** if and only if it is a topological embedding.

(Hint: to show that every topological embedding is a regular mono, make use of the 'indiscrete' topological space $\{0,1\}$ in which only $\emptyset$ and $\{0,1\}$ are open.)

Projective limits

The <u>projective limit</u> of an infinite sequence of maps $\left(X_{n+1} \xrightarrow{f_n} X_n\right)_{n\geq 0}$ in $\mathcal{C}$ is given by $P \in |\mathcal{C}|$ and $\left(P \xrightarrow{p_n} X_n\right)_{n\geq 0}$ such that:

- for all n, $p_n = f_n \circ p_{n+1}$; and
- for any family $\left(Z \xrightarrow{x_n} X_n\right)_{n\geq 0}$ such that $x_n = f_n \circ x_{n+1}$ $\forall n$, there exists a unique $Z \xrightarrow{z} P$ such that $p_n \circ z = x_n$ $\forall n$.

$$\begin{tikzcd}
\cdots \arrow[r, "f_3"'] & X_3 \arrow[r, "f_2"'] & X_2 \arrow[r, "f_1"'] & X_1 \arrow[r, "f_0"'] & X_0 \\
& Z \arrow[u, "x_3"] \arrow[ur, "x_2"] \arrow[urr, "x_1"] \arrow[urrr, "x_0"] \arrow[rr, dashed, "z"'] & & P \arrow[ull, "p_3"] \arrow[ul, "p_2"] \arrow[u, "p_1"] \arrow[ur, "p_0"] &
\end{tikzcd}$$

<u>Set</u> has projective limits

$$P := \left\{ (x_n)_{n\geq 0} \mid \forall n \ f_n(x_{n+1}) = x_n \right\}$$

$$p_n := (x_n)_{n\geq 0} \mapsto x_n \quad : P \to X_n$$

All notions we have seen today are instances of a general notion of <u>limit over a diagram</u>

A diagram has a shape given by a <u>graph</u>

Pullback	$\bullet \longrightarrow \bullet \longleftarrow \bullet$
Binary product	$\bullet \quad \bullet$
I-indexed product	$\overbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet}^{I}$
Terminal object	the empty graph
Equaliser	$\bullet \rightrightarrows \bullet$
Projective limit	$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$

Note that these graphs are <u>directed</u> and they can have multiple edges between the same two vertices. They are sometimes accordingly called <u>multidigraphs</u> (a.k.a. <u>quivers</u>)

A graph (as we shall call it for convenience; more precisely multidigraph or quiver) G has a collection $|G|$ of vertices, and for each $x, y \in |G|$ a collection $G(x,y)$ of edges with source x and target y.

G is locally small if every $G(x,y)$ is a set.

Similarly define locally finite & finite

G is small if it is locally small and $|G|$ is a set.

A graph homomorphism $H : G \to G'$ is given by:

- a function $H_0 : |G| \to |G'|$
- for every $x, y \in |G|$ a function $H_1 : G(x,y) \to G(H_0 x, H_0 y)$

Graph := category of small graphs and graph morphisms.

N.B. • A category is a graph with additional structure (identities + composition)

• Every functor is a graph morphism.

• These observations give a forgetful functor $\mathbf{Cat} \to \mathbf{Graph}$

Let G be a graph and $\mathcal{C}$ a category.

A <u>G-diagram</u> in $\mathcal{C}$ is a graph morphism $D : G \to \mathcal{C}$

The diagram D is <u>small</u> if G is small
<u>finite</u> finite

A <u>D-cone</u> $(Z, (\Gamma_u)_{u \in |G|})$

- an object $Z \in |\mathcal{C}|$, and
- a family $(Z \xrightarrow{\Gamma_u} Du)_{u \in |G|}$ of maps in $\mathcal{C}$

such that, for every edge $e \in D(u,v)$ in $\mathcal{C}$,

$De \circ \Gamma_u = \Gamma_v$

$$\begin{tikzcd} & Z \arrow[dl, "\Gamma_u"'] \arrow[dr, "\Gamma_v"] & \\ Du \arrow[rr, "De"'] & & Dv \end{tikzcd}$$

A <u>limit D-cone</u> is a D-cone $(L, (L \xrightarrow{\rho_u} Du)_{u \in |G|})$ such that, for every D-cone $(Z, (\Gamma_u)_{u \in |G|})$, there exists a unique $Z \xrightarrow{z} L$ such that $\rho_u \circ z = \Gamma_u$ $\forall u \in |G|$.

A category $\mathcal{C}$ is said to be <u>(small-) complete</u> if every small diagram in $\mathcal{C}$ has a limit cone.

<u>Theorem</u> The following are equivalent.

(1) $\mathcal{C}$ is complete

(2) $\mathcal{C}$ has products and equalisers

A category is said to be <u>finitely complete</u> if every finite diagram has a limit cone.

<u>Theorem</u> The following are equivalent

(1) $\mathcal{C}$ is finitely complete

(2) $\mathcal{C}$ has finite products and equalisers

(3) $\mathcal{C}$ has terminal object and pullbacks

Week 5 puzzle

The category <u>Rel</u> of relations has products. Find a concrete description of products in <u>Rel</u>.

Category Theory 2022-23

Lecture 6

11th November 2022

Reformulation of limit

The category of D-cones in $\mathcal{C}$ has

- Objects : D-cones
- Morphisms from $\left(Z', \left(Z' \xrightarrow{r'_u} Du\right)_{u \in |G|}\right)$
  to $\left(Z, \left(Z \xrightarrow{r_u} Du\right)_{u \in |G|}\right)$

  are maps $Z' \xrightarrow{h} Z$ in $\mathcal{C}$ s.t. $r_u \circ h = r'_u \ \forall u \in |G|$

A limit D-cone is then just a terminal object in the category of D-cones.

Limit D-cones are determined up to isomorphism, because terminal objects are determined up to isomorphism!

Projective limits

The <u>projective limit</u> of an infinite sequence of maps $(X_{n+1} \xrightarrow{f_n} X_n)_{n\geq 0}$ in $\mathcal{C}$ is given by $P \in |\mathcal{C}|$ and $(P \xrightarrow{p_n} X_n)_{n\geq 0}$ such that:

- for all n, $p_n = f_n \circ p_{n+1}$; and
- for any family $(Z \xrightarrow{x_n} X_n)_{n\geq 0}$ such that $x_n = f_n \circ x_{n+1}$ $\forall n$, there exists a unique $Z \xrightarrow{z} P$ such that $p_n \circ z = x_n$ $\forall n$.

$$\begin{tikzcd}
\cdots \arrow[r, "f_3"'] & X_3 \arrow[r, "f_2"'] & X_2 \arrow[r, "f_1"'] & X_1 \arrow[r, "f_0"'] & X_0 \\
 & Z \arrow[u, "x_3"'] \arrow[ur, "x_2"'] \arrow[urr, "x_1"'] \arrow[urrr, "x_0"'] \arrow[rr, dashed, "z"'] & & P \arrow[ull, "p_3"] \arrow[ul, "p_2"] \arrow[u, "p_1"] \arrow[ur, "p_0"']
\end{tikzcd}$$

Set has projective limits

$$P := \{ (x_n)_{n\geq 0} \mid \forall n \ f_n(x_{n+1}) = x_n \}$$

$$p_n := (x_n)_{n\geq 0} \mapsto x_n \ : P \to X_n$$

A category $\mathcal{C}$ is said to be <u>(small-) complete</u> if every small diagram in $\mathcal{C}$ has a limit cone.

<u>Theorem</u> The following are equivalent.

(1) $\mathcal{C}$ is complete

(2) $\mathcal{C}$ has products and equalisers

<u>Proof idea</u>

(1) $\Rightarrow$ (2) is trivial.

For (2) $\Rightarrow$ (1) let $D : G \to \mathcal{C}$ be a diagram

The limit cone is the components of the equaliser of

$$\prod_{u \in |G|} Du \underset{l_2}{\overset{l_1}{\rightrightarrows}} \prod_{(u,v,e) \in \{(u,v,e) \mid u,v \in |G|,\ e \in G(u,v)\}} Dv$$

where $l_1 = (De \circ \pi_u)_{(u,v,e)}$ and $l_2 = (\pi_v)_{(u,v,e)}$ $\square$

A category is said to be <u>finitely complete</u> if every finite diagram has a limit cone.

<u>Theorem</u> The following are equivalent

(1) $\mathcal{C}$ is finitely complete

(2) $\mathcal{C}$ has finite products and equalisers

(3) $\mathcal{C}$ has terminal object and pullbacks

<u>Proof idea</u> $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are immediate.

$(2) \Rightarrow (1)$ is proved in same way as last theorem.

For $(3) \Rightarrow (2)$, first construct the binary product $X \times Y$ as the pullback of $X \xrightarrow{!} 1 \xleftarrow{!} Y$.

Then construct the equaliser of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ as the pullback of $X \xrightarrow{(f,g)} Y \times Y \xleftarrow{(1_Y, 1_Y)} Y$.

$\square$

A <u>colimit</u> in C of a diagram $D : G \to C$ is a limit in C^{op} of $D : G^{op} \to C^{op}$

Type of colimit	Diagram shape
I-indexed coproducts	$\overbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet}^{I}$
Initial object	$\emptyset$
Coequaliser	$\bullet \rightrightarrows \bullet$
Pushout	$\bullet \leftarrow \bullet \rightarrow \bullet$
Direct limit e.g.	$\bullet \to \bullet \to \bullet \to \bullet \to \cdots$

$\mathcal{C}$ is <u>finitely cocomplete</u> if every finite diagram has a colimit.

$\mathcal{C}$ is <u>small cocomplete</u> (or just <u>cocomplete</u>) if every small diagram has a colimit.

<u>Theorem</u> The following are equivalent

- $\mathcal{C}$ is finitely cocomplete
- $\mathcal{C}$ has finite coproducts and coequalisers
- $\mathcal{C}$ has initial object and pushouts

<u>Theorem</u> The following are equivalent

- $\mathcal{C}$ is cocomplete
- $\mathcal{C}$ has coproducts and coequalisers

Both theorems are immediate from the corresponding theorems about limits <u>by duality</u>

Coproducts in Set

The coproduct of $(X_i)_{i\in I}$ is $\sum_{i\in I} X_i := \{(i,x) \mid i\in I, x\in X_i\}$

together with the cocone

$$\left(X_i \xrightarrow{in_i} \sum_{i\in I} X_i \right)_{i\in I}$$

$$in_i := x \mapsto (i,x)$$

Universal property

For any cocone $(X_i \xrightarrow{f_i} Z)_{i\in I}$

there exists a unique $\sum_{i\in I} X_i \xrightarrow{h} Z$ such that

$$\begin{tikzcd}
\sum_{i\in I} X_i \arrow[r, "h"] & Z \\
X_i \arrow[u, "in_i"] \arrow[ur, "f_i"'] &
\end{tikzcd}$$

commutes $\forall i\in I$

namely $h := (i,x) \mapsto f_i(x)$.

In $\underline{\text{Vect}}_K$ finite coproducts coincide with finite products. ($\underline{\text{Vect}}_K$ has bi<u>products</u>.)

Given vector spaces V, W, the coproduct is

$$V \xrightarrow{v \mapsto (v,0)} V \times W \xleftarrow{(0,w) \leftarrow\!\shortmid\, w} W$$

Indeed given linear $V \xrightarrow{f} U$ and $W \xrightarrow{g} U$ the map $h := (v,w) \longmapsto f(v) + g(w)$ is the unique linear map such that the diagram below commutes

$$\begin{tikzcd}
V \arrow[r, "{(-,0)}"] \arrow[dr, "f"'] & V \times W \arrow[d, "h"] & W \arrow[l, "{(0,-)}"'] \arrow[dl, "g"] \\
& U &
\end{tikzcd}$$

A <u>coequaliser</u> is a colimit for

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

i.e., an object Q together with $Y \xrightarrow{q} Q$ s.t. $q \circ f = q \circ g$

satisfying:

for any $Y \xrightarrow{z} Z$ s.t. $z \circ f = z \circ g$

there exists a unique $Q \xrightarrow{w} Z$ s.t. $z = w \circ q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, "w"] \\ & & Z \end{tikzcd}$$

Every q arising as a coequaliser is epi.

Morphisms that arise as coequalisers are called <u>regular epis</u>.

Coequalisers in <u>Set</u>

The coequaliser in <u>Set</u> of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

is $Y \xrightarrow{q} Q$ defined as follows

$$Q := Y/\sim$$

where $\sim$ is the smallest equivalence relation such that $f(x) \sim g(x)$, for every $x \in X$.

$$q := y \mapsto [y]_\sim$$

(every element is mapped to its equivalence class)

Relations in a category

A representative for a <u>relation</u> between X and Y in C is a jointly monic pair

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[dr, "r_2"'] & X \\ & Y \end{tikzcd}$$

Recall <u>joint monicity</u> means, for any $Z \underset{y}{\overset{x}{\rightrightarrows}} R$

if $r_1 \circ x = r_1 \circ y$ and $r_2 \circ x = r_2 \circ y$ then $x = y$.

(If C has products this is $\equiv$ to $R \xrightarrow{(r_1, r_2)} X \times Y$ is mono.)

Given relations $\begin{tikzcd}[row sep=small] R \arrow[r, "r_1"] \arrow[dr, "r_2"'] & X \\ & Y \end{tikzcd}$ and $\begin{tikzcd}[row sep=small] R' \arrow[r, "r_1'"] \arrow[dr, "r_2'"'] & X \\ & Y \end{tikzcd}$

$R \sqsubseteq R'$ means there exists a (necessarily unique) diagonal making the two triangles commute

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, dashed] & X \\ Y & R' \arrow[l, "r_2'"] \arrow[u, "r_1'"'] \end{tikzcd}$$

$R \equiv R'$ means $R \sqsubseteq R'$ and $R' \sqsubseteq R$

Given $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} \begin{matrix} X \\ Y \end{matrix}$

we say $Z \overset{x}{\underset{y}{\rightrightarrows}} \begin{matrix} X \\ Y \end{matrix}$ __are related by__ R $\quad (xRy)$

if there exists a (necessarily unique) commuting diagonal

$$\begin{tikzcd} Z \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, dashed] & X \\ Y & R \arrow[l, "r_2"] \arrow[u, "r_1"'] \end{tikzcd}$$

$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} \begin{matrix} X \\ X \end{matrix}$ is an __equivalence relation__ if

$\forall Z \quad \forall Z \xrightarrow{x} X \qquad xRx$

$\forall Z \ \forall x, y : Z \to X \quad xRy \Rightarrow yRx$

$\forall Z \ \forall x, y, z : Z \to X \quad xRy \wedge yRz \Rightarrow xRz$

The <u>kernel pair</u> of $X \xrightarrow{f} Y$ is the pullback $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X$ of f along itself

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

Exercise

1) f is mono $\Leftrightarrow$ it has $X \underset{1_X}{\overset{1_X}{\rightrightarrows}} X$ as its kernel pair.

2) Any kernel pair is an equivalence relation

The symbiotic relationship between <u>coequalisers and kernel pairs</u>

1) If $Y \xrightarrow{q} Q$ is regular epi and q has a kernel pair $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} Q$ then q is the coequaliser of r_1, r_2

2) If $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} Y$ is a kernel pair and it has a coequaliser $Y \xrightarrow{q} Q$ then $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} Y$ is the kernel pair of q

A diagram

$$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} Y \xrightarrow{q} Q$$

in which r_1, r_2 is the kernel pair of q and q is the coequaliser of r_1, r_2 is called an <u>exact fork</u>.

Category Theory 2022-23

Lecture 7

18th November 2022

<u>Monoidal structure</u> on $\mathcal{C}$

is given by

- An object $I \in |\mathcal{C}|$
- A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- Natural isomorphisms

$$X \otimes (Y \otimes Z) \xrightarrow{\alpha_{XYZ}} (X \otimes Y) \otimes Z$$

$$I \otimes X \xrightarrow{\lambda_X} X \qquad X \otimes I \xrightarrow{\rho_X} X$$

Such that the following equalities hold

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes W)) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \alpha"'] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\alpha"] & ((X \otimes Y) \otimes Z) \otimes W \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\alpha"] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[u, "\alpha \otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (I \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \lambda"'] & & (X \otimes I) \otimes Y \arrow[dl, "\rho \otimes 1"] \\
& X \otimes Y &
\end{tikzcd}$$

$$I \otimes I \xrightarrow{\lambda} I \;=\; I \otimes I \xrightarrow{\rho} I$$

We say that $\mathcal{C}$ is a <u>monoidal category</u>

If $\mathcal{C}$ is a monoidal category then the same $\otimes, I$ define monoidal structure on $\mathcal{C}^{op}$. (Use $\alpha^{-1}, \rho^{-1}, \lambda^{-1}$.)

If $\mathcal{C}$ is a monoidal category then the same structure exhibits $\mathcal{C}_{iso}$ as a monoidal category. ($\mathcal{C}_{iso}$ is the category with $|\mathcal{C}_{iso}| = |\mathcal{C}|$, whose morphisms are the isos in $\mathcal{C}$.)

Symmetric monoidal structure is given by monoidal structure together with a natural isomorphism

$$X \otimes Y \xrightarrow{\sigma_{XY}} Y \otimes X$$

satisfying

$$\begin{tikzcd} X \otimes Y \arrow[rr, "1"] \arrow[dr, "\sigma"'] & & X \otimes Y \\ & Y \otimes X \arrow[ur, "\sigma"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \sigma"'] & (X \otimes Y) \otimes Z \arrow[r, "\sigma"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\ X \otimes (Z \otimes Y) \arrow[r, "\alpha"] & (X \otimes Z) \otimes Y \arrow[r, "\sigma \otimes 1"] & (Z \otimes X) \otimes Y \end{tikzcd}$$

$$\begin{tikzcd} X \otimes I \arrow[rr, "\sigma"] \arrow[dr, "\rho"'] & & I \otimes X \arrow[dl, "\lambda"] \\ & X & \end{tikzcd}$$

We say that $\mathcal{C}$ is a **symmetric monoidal category** (smc)

Again, the same structure exhibits $\mathcal{C}^{op}$ as an smc.

Finite products

Binary product defines a functor $(-)\times(-) : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$

$$\begin{array}{ccc} X & & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & & Y' \end{array} \quad \longmapsto \quad \begin{array}{c} X\times Y \\ \downarrow f\times g := (f\circ\pi_1,\ g\circ\pi_2) \\ X'\times Y' \end{array}$$

Define $I := 1$ (terminal object)

$$X\times(Y\times Z) \xrightarrow{\alpha} (X\times Y)\times Z \quad \text{given by } \alpha := ((\pi_1, \pi_1\circ\pi_2), \pi_2\circ\pi_2)$$

$$1\times X \xrightarrow{\lambda} X \qquad \lambda := \pi_2$$

$$X\times 1 \xrightarrow{\rho} X \qquad \rho := \pi_1$$

This is <u>symmetric</u> monoidal structure

$$X\times Y \xrightarrow{\sigma} Y\times X \qquad \sigma := (\pi_2, \pi_1)$$

Any category with finite products is a symmetric monoidal category

By duality, so is any category with finite coproducts.

A category can carry more than one (symmetric) monoidal structure.

Rel

The set-theoretic coproduct $X+Y$ is both coproduct and product in the category <u>Rel</u>.

[cf. Week 5 puzzle.]

So this is one (symmetric) monoidal structure on <u>Rel</u>.

Although the set-theoretic product $X \times Y$ is not the product in <u>Rel</u> it is a symmetric monoidal product. E.g., the functorial action is given by

$$\begin{array}{ccc} X & & X' \\ R\downarrow & & \downarrow R' \\ Y & & Y' \end{array} \quad \longmapsto \quad \begin{array}{c} X \times X' \\ \downarrow R \times R' \\ Y \times Y' \end{array}$$

$$(x, x') \, (R \times R') \, (y, y') \iff x R y \text{ and } x' R' y'.$$

$[\mathcal{C}, \mathcal{C}]$ – the category of <u>endofunctors</u> on $\mathcal{C}$.

$$G \otimes F := GF \qquad \text{(composition)}$$
$$I := 1_{\mathcal{C}}$$

This defines <u>strict</u> monoidal structure, all α, λ, ρ maps are identities.

This monoidal structure is <u>not symmetric</u>

$\underline{\mathrm{Mat}_K}$

Define $m \otimes n := mn$

Given $n \xrightarrow{A} m$ $n' \xrightarrow{B} m'$ define $n \otimes n' \xrightarrow{A \otimes B} mm'$

to be the $mm' \times nn'$ matrix

$$\begin{bmatrix} (a_{11} B) & \cdots & (a_{1n} B) \\ \vdots & & \vdots \\ (a_{m1} B) & & (a_{mn} B) \end{bmatrix}$$

i.e C where $C_{(i-1)m'+i', (j-1)n'+j'} = a_{ij} \cdot b_{i'j'}$ for $\begin{matrix} 1 \leq i \leq m \\ 1 \leq i' \leq m' \\ 1 \leq j \leq n \\ 1 \leq j' \leq n' \end{matrix}$

$I := 1$

ρ and λ are easy.

Have fun working out α!

This is <u>symmetric</u> monoidal structure

$m \otimes n \xrightarrow{\sigma} n \otimes m$ is the $(mn \times mn)$ square matrix:

$$\sigma_{d,e} = \begin{cases} 1 & \text{if } \exists\, 1 \leq i \leq m \;\; 1 \leq j \leq n \text{ s.t. } \begin{matrix} d = (i-1)m + j \\ e = i + (j-1)n \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

More generally $\underline{\mathrm{Vect}}_K$

The <u>tensor product</u> $V \otimes W$ of vector spaces enjoys the following characterising property

There is a bilinear map

$$\psi_{VW} : V \times W \longrightarrow V \otimes W$$

Such that, for any vector space U and bilinear $f : V \times W \to U$, there exists a unique linear map $g : V \otimes W \to U$ satisfying $\otimes$

$$\begin{tikzcd} V \otimes W \arrow[r, "g"] & U \\ V \times W \arrow[u, "\psi"] \arrow[ur, "f"'] & \end{tikzcd}$$

N.B. This is <u>not</u> a diagram in $\underline{\mathrm{Vect}}_K$!

So linear maps $V \otimes W \to U$ are in 1-1-correspondence with bilinear maps $V \times W \to U$

Define $I := K$

Then $\otimes, I$ endow $\underline{\mathrm{Vect}}_K$ with symmetric monoidal structure.

<u>Exercise</u> Work out the details, either abstractly using $\otimes$ or concretely using an explicit construction of $V \otimes W$ (e.g. as a quotient space).

Monoidal closed structure

A monoidal category $\mathcal{C}$ is <u>(left) closed</u> if, for every $X, Y \in |\mathcal{C}|$, there is an object $[X,Y]$ and map

$$[X,Y] \otimes X \xrightarrow{ev_{X,Y}} Y$$

such that, for every map $Z \otimes X \xrightarrow{f} Y$, there exists a unique map $Z \xrightarrow{\Lambda f} [X,Y]$ such that

$$\begin{tikzcd}
{[X,Y] \otimes X} \arrow[r, "ev"] & Y \\
Z \otimes X \arrow[u, "\Lambda f \otimes 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

There is also a notion of <u>right closed</u> using maps

$$X \otimes [X,Y]^R \longrightarrow Y$$

In the case of a symmetric monoidal category the notions of left and right closed coincide, and one simply says <u>closed</u>

A category is <u>Cartesian closed</u> if it has finite products and the product monoidal structure is closed.

<u>Set</u> is cartesian closed

Define $[X, Y] := Y^X$ (set of all functions $X \to Y$)

$$ev_{X,Y} := (f, x) \mapsto f(x) : Y^X \times X \to Y$$

Given $Z \times X \xrightarrow{f} Y$, we must show there is a unique $Z \xrightarrow{\Lambda f} Y^X$ such that the diagram below commutes

$$\begin{tikzcd} Y^X \times X \arrow[r, "ev"] & Y \\ Z \times X \arrow[u, "\Lambda_f \times 1_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

A

This diagram says

$$\begin{aligned} f(x,z) &= (ev \circ (\Lambda_f \times 1_x))(z,x) \\ &= ev(\Lambda_f(z), x) \\ &= \Lambda_f(z)(x) \end{aligned}$$

So the function

$$\Lambda_f := z \mapsto (x \mapsto f(z,x)) : Z \to Y^X$$

is uniquely determined by the commutivity.

$\underline{\text{Vect}}_k$ is monoidal closed

Define $[V, W] := V \multimap W$ the vector space of all linear functions from V to W

Notice that the evaluation function

$$f, v \mapsto f(v) \quad : \quad (V \multimap W) \times V \longrightarrow W$$

is bilinear. So it corresponds to a unique map

$$(V \multimap W) \otimes V \xrightarrow{ev} W$$

There is also a 1-1-correspondence

$$\frac{\text{bilinear } [U \times V, W]}{\text{linear } [U, V \multimap W]} \qquad \downarrow \uparrow$$

$$f : U \times V \to W \quad \mapsto \quad (u \mapsto (v \mapsto f(u, v)))$$

$$g : U \to (V \multimap W) \quad \mapsto \quad ((u, v) \mapsto g(u)(v))$$

This gives us Λ viz:

$$\underline{\text{Vect}}_k (U \times V, W) \underset{\cong}{\xrightarrow{\Lambda}} \underline{\text{Vect}}_k (U, V \multimap W)$$

Week 7 puzzle

Consider the functor categories

$[\underline{G}, \underline{Set}]$

$[\underline{\Delta}, \underline{Set}]$

from Week 3. Both categories are cartesian closed. Find explicit descriptions of the closed structure

Category Theory 2022-23

Lecture 8

25th November 2022

Monoidal closed structure

Definition (Monoidal closure)

A monoidal category C is <u>(left) closed</u> if, for every $X, Y \in |C|$, there is an object $[X,Y]$ and map

$$[X,Y] \otimes X \xrightarrow{ev_{X,Y}} Y$$

such that, for every map $Z \otimes X \xrightarrow{f} Y$, there exists a unique map $Z \xrightarrow{\Lambda f} [X,Y]$ such that

$$\begin{tikzcd}
{[X,Y] \otimes X} \arrow[r, "ev"] & Y \\
Z \otimes X \arrow[u, "\Lambda f \otimes 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proposition (alternative formulation of monoidal closure)

A monoidal category is (left) closed if and only if, for every object X, there is a functor $[X,-] : C \to C$ together with a natural (in Z and Y) bijection

$$\Lambda : C(Z \otimes X, Y) \xrightarrow[\cong]{} C(Z, [X,Y])$$

The <u>free vector space</u> $\underline{V}(X)$ over a set X is defined by (Fill in the obvious definitions of addition and scalar multiplication)

$$\underline{V}(X) := \text{functions } X \to K \text{ with finite support}$$

($f : X \to K$ has <u>finite support</u> if $\{x \in X \mid f(x) \neq 0\}$ is finite)

This is characterised up to linear isomorphism by the following <u>universal property</u>

For any set X, vector space W and function $f : X \to W$, there exists a unique linear $h : \underline{V}(X) \to W$ such that

$$\begin{tikzcd} \underline{V}(X) \arrow[r, "h"] & W \\ X \arrow[u, "\delta"] \arrow[ur, "f"'] & \end{tikzcd}$$

(A diagram in <u>Set</u>)

where $\delta_x(y) := \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$

A more pedantic formulation of the same universal property making the role of the forgetful functor $\underline{U} : \underline{\text{Vect}} \to \underline{\text{Set}}$ explicit.

For any set X, vector space W and map $X \xrightarrow{f} \underline{U}W$ in $\underline{\text{Set}}$, there exists a unique map $\underline{V}(X) \xrightarrow{h} W$ in $\underline{\text{Vect}}$ such that

$$
\begin{tikzcd}
\underline{U}(\underline{V}(X)) \arrow[r, "\underline{U}h"] & \underline{U}W \\
X \arrow[u, "\delta"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

(A)

An alternative equivalent statement

There is a functor $\underline{V} : \underline{\text{Set}} \to \underline{\text{Vect}}$ together with natural (in X and W) bijections

$$\psi_{X,W} : \underline{\text{Vect}}(\underline{V}X, W) \xrightarrow{\cong} \underline{\text{Set}}(X, \underline{U}W)$$

Definition of adjunction

An <u>adjunction</u> (F, G, ψ) between $\mathcal{C}$ and $\mathcal{D}$ is given by

- Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$
- Natural (in $X \in |\mathcal{C}|$ and $Y \in |\mathcal{D}|$) bijections

$$\psi_{X,Y} : \mathcal{D}(FX, Y) \xrightarrow{\cong} \mathcal{C}(X, GY)$$

We say that F is <u>left adjoint</u> to G

G <u>right adjoint</u> F

and we write $F \dashv G : \mathcal{D} \to \mathcal{C}$ or $F \dashv G$.

Theorem (Equivalent formulations)

Each of the 3 sets of information below is equivalent to specifying an adjunction (F, G, ψ) between $\mathcal{C}$ and $\mathcal{D}$.

① A functor $G : \mathcal{D} \to \mathcal{C}$, function $F : |\mathcal{C}| \to |\mathcal{D}|$ and family $(X \xrightarrow{\eta_X} GFX)_{X \in |\mathcal{C}|}$ such that, for any $X \in |\mathcal{C}|$, $Y \in |\mathcal{D}|$ and $X \xrightarrow{f} GY$ in $\mathcal{C}$, there exists a unique $FX \xrightarrow{g} Y$ in $\mathcal{D}$ s.t.

$$\begin{tikzcd} GFX \arrow[r, "Gg"] & GY \\ X \arrow[u, "\eta_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes in $\mathcal{C}$.

② A functor $F : \mathcal{C} \to \mathcal{D}$, function $G : |\mathcal{D}| \to |\mathcal{C}|$ and family $(FGY \xrightarrow{\varepsilon_Y} Y)_{Y \in |\mathcal{D}|}$ such that, for any $X \in |\mathcal{C}|$, $Y \in |\mathcal{D}|$ and $FX \xrightarrow{g} Y$ in $\mathcal{D}$, there exists a unique $X \xrightarrow{f} GY$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[dr, "g"'] & FGY \arrow[d, "\varepsilon_Y"] \\ & Y \end{tikzcd}$$

commutes in $\mathcal{D}$.

③ Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, and natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ s.t.

$$\begin{tikzcd} FGF \arrow[r, Rightarrow, "\varepsilon F"] & F \\ F \arrow[u, Rightarrow, "F\eta"] \arrow[ur, Rightarrow, "1_F"'] & \end{tikzcd} \qquad \begin{tikzcd} GFG \arrow[r, Rightarrow, "G\varepsilon"] & G \\ G \arrow[u, Rightarrow, "\eta G"] \arrow[ur, Rightarrow, "1_G"'] & \end{tikzcd}$$

Proof outline that (1) is equivalent to an adjunction

From adjunction to (1)

Let (F, G, ψ) be an adjunction.

Thus $G: \mathcal{D} \to \mathcal{C}$ and $F: |\mathcal{C}| \to |\mathcal{D}|$ are given

Consider $\psi_{X,FX} : \mathcal{D}(FX, FX) \xrightarrow{\cong} \mathcal{C}(X, GFX)$

and define $X \xrightarrow{\eta_X} GFX := \psi(1_{FX})$.

The naturality of $\psi_{X,Y}$ in Y gives us for any map $FX \xrightarrow{g} Y$ in $\mathcal{D}$

$$\begin{tikzcd} \mathcal{D}(FX, FX) \arrow[r, "\psi_{X,FX}"] \arrow[d, "\mathcal{C}(FX,g)"'] & \mathcal{C}(X, GFX) \arrow[d, "\mathcal{C}(X,Gg)"] \\ \mathcal{D}(FX, Y) \arrow[r, "\psi_{X,Y}"] & \mathcal{C}(X, GY) \end{tikzcd}$$

$i \mapsto g \circ i$ $\qquad$ $h \mapsto Gg \circ h$

In particular $\psi_{X,Y}(g) = \psi_{X,Y}(\mathcal{C}(FX, g)(1_{FX}))$
$= \mathcal{C}(X, Gg)(\psi_{X,FX}(1_{FX})) = Gg \circ \eta_X$.

Since $\psi_{X,Y}$ is a bijection, for any $X \xrightarrow{f} GY$ in $\mathcal{C}$,
$\psi^{-1}(f)$ is the unique g s.t. $f = Gg \circ \eta_X$.

From ① to adjunction.

Suppose we have $G : \mathcal{D} \to \mathcal{C}$, $F : |\mathcal{C}| \to |\mathcal{D}|$ and $(X \xrightarrow{\eta_X} GFX)_{X \in |\mathcal{C}|}$ as in ①.

The functorial action of F is

$$\begin{array}{c} X \\ f \downarrow \\ X' \end{array} \longmapsto \text{the unique } FX \xrightarrow{g} FX' \text{ such that } \begin{tikzcd} GFX \arrow[r, "Gg"] & GFX' \\ X \arrow[u, "\eta_X"] \arrow[r, "f"'] & X' \arrow[u, "\eta_{X'}"'] \end{tikzcd}$$

The bijection $\psi_{X,Y} : \mathcal{D}(FX, Y) \xrightarrow{\cong} \mathcal{C}(X, GY)$ is defined by

$$\psi(FX \xrightarrow{g} Y) := X \xrightarrow{\eta_X} GFX \xrightarrow{Gg} GY$$

The diagrammatic property in ① says that this is indeed a bijection.

One then verifies that the functorial action of F preserves identities and composition and the naturality of $\psi_{X,Y}$. $\square$

Examples of adjunctions

The free-vector-space functor $\underline{V} : \underline{Set} \to \underline{Vect}$ is left adjoint to the forgetful $\underline{U} : \underline{Vect} \longrightarrow \underline{Set}$.

(Property (A) is formulation (1) of an adjunction.)

The free-group functor $F : \underline{Set} \to \underline{Grp}$ is left adjoint to the forgetful $\underline{U} : \underline{Grp} \to \underline{Set}$.

(Similar statements hold for other free-algebra functors.)

A monoidal category $\mathcal{C}$ is left closed iff for every object X, the functor $(-) \otimes X : \mathcal{C} \to \mathcal{C}$ has a right adjoint. (The right adjoint is $[X, -]$.)

(The definition of left closure is formulation (2) of adjunction.)

Any equivalence of categories (F, G, α, β) is an adjunction with $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $G \dashv F : \mathcal{C} \to \mathcal{D}$.

(Use formulation (3), exploiting α^{-1} and β^{-1} as appropriate.)

Consider $(-)^* : \underline{\mathrm{Vect}}^{op} \to \underline{\mathrm{Vect}}$, the contravariant functor mapping every vector space to its dual.

Then $(-)^*$ is <u>self adjoint</u>: $(-)^* \dashv (-)^* : \underline{\mathrm{Vect}}^{op} \to \underline{\mathrm{Vect}}$

(There are natural bijections

$$\underline{\mathrm{Vect}}(V, W^*) \cong \underline{\mathrm{Vect}}(V \otimes W, k)$$
$$\cong \mathrm{Vect}(W, V^*) \cong \underline{\mathrm{Vect}}^{op}(V^*, W) \ .)$$

More generally, if C is symm. mon. closed then, for every $Y \in |C|$, $[-, Y]$ extends to a functor $[-, Y] : C^{op} \to C$ that is self adjoint $[-, Y] \dashv [-, Y] : C^{op} \to C$

<u>Exercise</u>: • verify this last point.

• Generalise the statement to categories with (non-symmetric) monoidal structure that are both left and right closed.

Proposition (Adjoints determined up to isomorphism)

Suppose $F \dashv G : \mathcal{D} \to \mathcal{C}$.

① If $F' \dashv G : \mathcal{D} \to \mathcal{C}$ then $F' \cong F : \mathcal{C} \to \mathcal{D}$.

② If $F \dashv G' : \mathcal{D} \to \mathcal{C}$ then $G' \cong G : \mathcal{D} \to \mathcal{C}$

Proof

① We use reformulation ① of adjunction.

The adjunctions $F \dashv G$ and $F' \dashv G$ give us $X \xrightarrow{\varepsilon_X} GFX$ and $X \xrightarrow{\varepsilon'_X} GF'X$.

Applying reformulation ① we get $FX \xrightarrow{h_X} F'X$ and $F'X \xrightarrow{h'_X} FX$

unique s.t.

$$\begin{tikzcd} GFX \arrow[r, "Gh_X"] & GF'X \\ X \arrow[u, "\varepsilon_X"] \arrow[ur, "\varepsilon'_X"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} GF'X \arrow[r, "Gh'_X"] & GFX \\ X \arrow[u, "\varepsilon'_X"] \arrow[ur, "\varepsilon_X"'] & \end{tikzcd}$$

It is easy to show that their composites are identities and that $(h_X)_X$ and $(h'_X)_X$ are natural.

② Follows from ① because $F \dashv G : \mathcal{D} \to \mathcal{C}$ iff $G \dashv F : \mathcal{C}^{op} \to \mathcal{D}^{op}$.

(This 'iff' is immediate from the definition of adjunction.)

Proposition (Composition of adjunctions)

If $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{E} \to \mathcal{D}$ then $F'F \dashv GG' : \mathcal{E} \to \mathcal{C}$

Proof $\mathcal{C}(X, GG'Z) \cong \mathcal{D}(FX, G'Z) \cong \mathcal{E}(F'FX, Z)$.

Naturality holds because the composition of natural bijections preserves it. $\square$

Definition (Preservation of limits)

A functor $F : C \to C'$ preserves the limit of a diagram $D\ G \to C$ (G a graph) if, for any limit cone $(L, (L \xrightarrow{\beta_x} Dx)_{x \in G})$, it holds that $(FL, (FL \xrightarrow{F\beta_x} FDx)_{x \in G})$ is a limit cone in C' for the diagram $FD : G \to C'$.

$$
\left(
\begin{tikzcd}
 & L \arrow[dl, "\beta"'] \arrow[d, "\beta_y"] \\
Dx \arrow[r, "De"] & Dy
\end{tikzcd}
\right)_{x,y,e}
\qquad
\left(
\begin{tikzcd}
 & FL \arrow[dl, "F\beta_x"'] \arrow[d, "F\beta_y"] \\
FDx \arrow[r, "FDe"] & FDy
\end{tikzcd}
\right)_{x,y,e}
$$

F is said to <u>preserve (existing) limits of shape G</u> if F preserves the limit of D, for every diagram $F : G \to C$ (that has a limit).

Instances of the above: F preserves (finite) products, F preserves pullbacks, F preserves equalisers, F preserves finite limits (a.k.a. is <u>left exact</u>) etc.

F is said to <u>preserve (existing) limits</u> if F preserves (existing) limits of shape G, for every graph G.

There are dual definitions of what it means for F to <u>preserve colimits</u>

Proposition Suppose $F' \dashv F : \mathcal{C} \to \mathcal{C}'$. Then F preserves limits and F' preserves colimits.

(Right adjoints preserve limits, left adjoints preserve colimits.)

Proof Let $(L, (L \xrightarrow{\rho_x} Dx)_{x \in |G|})$ be a limit cone for $D : G \to \mathcal{C}$.

We need to show $(FL, (FL \xrightarrow{F\rho_x} FDx)_x)$ is a limit for $FD : G \to \mathcal{C}'$.

Let $(Z, (Z \xrightarrow{\Gamma_x} FDx)_x)$ be a cone for FD.

Then $(F'Z, (F'Z \xrightarrow{\psi^{-1}\Gamma_x} Dx)_x)$ is a cone for D.

$$
\begin{tikzcd}
Z \arrow[rr, dotted, "\psi\lambda"] \arrow[d, "\Gamma_x"'] \arrow[drr, "\Gamma_y" near end] & & FL \arrow[d, "F\rho_y"] \arrow[dll, "F\rho_x"' near start] \\
\Big(FDx \arrow[rr, "FDe"] & & FDy \Big)_{x,y,e}
\end{tikzcd}
\qquad
\begin{tikzcd}
F'Z \arrow[rr, dotted, "\lambda"] \arrow[d, "\psi^{-1}\Gamma_x"'] \arrow[drr, "\psi^{-1}\Gamma_y" near end] & & L \arrow[d, "\rho_y"] \arrow[dll, "\rho_x"' near start] \\
\Big(Dx \arrow[rr, "De"] & & Dy \Big)_{x,y,e}
\end{tikzcd}
$$

Let $F'Z \xrightarrow{\lambda} L$ be the unique cone morphism in $\mathcal{C}$.

Then $Z \xrightarrow{\psi\lambda} FL$ is the unique cone morphism in $\mathcal{C}'$.

The statement for colimits follows by duality because $F' \dashv F : \mathcal{C} \to \mathcal{C}' \iff F' \dashv F : \mathcal{C}'^{op} \to \mathcal{C}^{op}$.

$\square$

Application The 'arithmetic' of exponentiation

Suppose throughout that $\mathcal{C}$ is left monoidal closed.

$$[I, X] \cong X \qquad \text{cf. } x^1 = x$$

$$[Y \otimes Z, X] \cong [Z, [Y, X]] \qquad x^{yz} = (x^y)^z$$

If $\mathcal{C}$ has finite products then

$$[X, 1] \cong 1 \qquad 1^x = 1$$

$$[X, Y \times Z] \cong [X, Y] \times [X, Z] \qquad (yz)^x = y^x z^x$$

If $\mathcal{C}$ has finite coproducts then

$$0 \otimes X \cong 0 \qquad 0 . x = 0$$

$$(Y + Z) \otimes X \cong (Y \otimes X) + (Z \otimes X) \qquad (y+z)x = yx + zx$$

If $\mathcal{C}$ has finite products and coproducts then

$$[0, X] \cong 1 \qquad x^0 = 1$$

$$[Y + Z, X] \cong [Y, X] \times [Z, X] \qquad x^{y+z} = x^y x^z$$

N.B., In the case of a cartesian closed category, $\otimes$ and $\times$ coincide.

Exercise Prove the above!

Category Theory 2022-23

Lecture 9

2nd December 2022

Monoids in Set

A monoid is a structure $(X, \bullet, e)$ where X is a set and $\bullet : X \times X \to X$ and $e \in X$ are such that

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \qquad \forall x y z \in X$$

$$x \bullet e = x = e \bullet x \qquad \forall x \in X$$

(N.B. given X and $\bullet$, e is uniquely detemined.)

A homomorphism of monoids from $(X, \bullet, e)$ to $(X', \bullet', e')$ is a function $h : X \to X'$ such that

$$h(x \bullet y) = h(x) \bullet' h(y)$$

$$h(e) = e'$$

The category Mon has monoids as objects and homomorphisms as morphisms.

Monoids in a category $\mathcal{C}$ with finite products

A *monoid* is a structure $(X, \bullet, e)$ where $X \in |\mathcal{C}|$

and $X \times X \xrightarrow{\bullet} X$ and $1 \xrightarrow{e} X$ are such that

$$\begin{tikzcd}
X \times (X \times X) \arrow[r, "1 \times \bullet"] \arrow[d, "\alpha_X"'] & X \times X \arrow[dr, "\bullet"] & \\
(X \times X) \times X \arrow[r, "\bullet \times 1_X"] & X \times X \arrow[r, "\bullet"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \times 1 \arrow[d, "1_X \times e"'] & X \arrow[l, "\rho_X^{-1}"'] \arrow[r, "\lambda_X^{-1}"] \arrow[d, "1_X"] & 1 \times X \arrow[d, "e \times 1_X"] \\
X \times X \arrow[r, "\bullet"'] & X & X \times X \arrow[l, "\bullet"]
\end{tikzcd}$$

(<u>Exercise</u> $1 \xrightarrow{e} X$ is uniquely determined by $X \times X \xrightarrow{\bullet} X$.)

A *homomorphism* from $(X, \bullet, e)$ to $(X', \bullet', e')$ is a map $X \xrightarrow{h} X'$ s.t.

$$\begin{tikzcd}
X \times X \arrow[r, "h \times h"] \arrow[d, "\bullet"'] & X' \times X' \arrow[d, "\bullet'"] \\
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
X \arrow[rr, "h"] & & X'
\end{tikzcd}$$

$\mathrm{Mon}_{\mathcal{C}}$: The category of monoids in $\mathcal{C}$ and homomorphisms

Groups in $\underline{\text{Set}}$

A *group* is a monoid $(X, \bullet, e)$ such that,
for every $x \in X$, there exists $x^{-1} \in X$ with $x \bullet x^{-1} = e = x^{-1} \bullet e$

(N.B. x^{-1} is uniquely determined.)

A *homomorphism* of groups from $(X, \bullet, e)$ to $(X', \bullet', e')$
is simply a monoid homomorphism $h : X \to X'$

(N.B. It follows that $h(x^{-1}) = h(x)^{-1}$ inverse in X inverse in X'

Also, the fact that $(X', \bullet', e')$ is a group means that
the equation $h(e) = e'$ follows from $h(x \bullet y) = h(x) \bullet' h(y)$ alone.)

$\underline{\text{Grp}}$ is the category of groups and homomorphisms

Groups in a category C with finite products

A *group* is a monoid $(X, X\times X \xrightarrow{\bullet} X, 1 \xrightarrow{e} X)$ for which there exists $X \xrightarrow{(\cdot)^{-1}} X$ such that

$$\begin{tikzcd} X\times X \arrow[d, "1\times(-)^{-1}"'] & X \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "!"] & X\times X \arrow[d, "(-)^{-1}\times 1"] \\ & 1 \arrow[d, "e"] & \\ X\times X \arrow[r, "\bullet"'] & X & X\times X \arrow[l, "\bullet"] \end{tikzcd}$$

$\Delta := X \xrightarrow{\langle 1_X, 1_X\rangle} X\times X$

(Exercise $X \xrightarrow{(-)^{-1}} X$ is uniquely determined by $X\times X \xrightarrow{\bullet} X$.)

A *homomorphism* of groups is just a homomorphism of monoids.

(Exercise For any monoid homomorphism $(X,\bullet,e) \xrightarrow{h} (X,\bullet,e)$ between groups it follows that

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "(-)^{-1}"'] & X \arrow[d, "(-)^{-1}"] \\ X \arrow[r, "h"] & X \end{tikzcd}$$

Moreover preservation of units is a consequence of preservation of multiplication.)

Grp_C is the category of groups in C and homomorphisms.

Grp_C is a full subcategory of Mon_C

C is a full subcategory of D if

$$|C| \subseteq |D|$$

and $\quad C(X,Y) = D(X,Y) \quad \forall X,Y \in |C|$

Examples

In **Set**: ordinary groups and their homomorphisms

In **Top**: topological groups and their continuous homomorphisms

(In a topological group, group multiplication

$$\cdot : X \times X \longrightarrow X$$

is <u>jointly continuous</u>; i.e continuous w.r.t the product topology, and the inverse function

$$(-)^{-1} : X \longrightarrow X$$

is continuous. These properties imply that X is Hausdorff (T_2).)

In **Man** (the category of smooth maps between differentiable manifolds): Lie groups and smooth homomorphisms.

Monoids in a category $\mathcal{C}$ with monoidal structure

A monoid is a structure $(X, \bullet, e)$ where $X \in |\mathcal{C}|$ and $X \otimes X \xrightarrow{\bullet} X$ and $I \xrightarrow{e} X$ are such that

$$\begin{tikzcd}
X \otimes (X \otimes X) \arrow[r, "1_X \otimes \bullet"] \arrow[d, "\alpha_X"'] & X \otimes X \arrow[dr, "\bullet"] & \\
 & & X \\
(X \otimes X) \otimes X \arrow[r, "\bullet \otimes 1_X"] & X \otimes X \arrow[ur, "\bullet"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes I \arrow[d, "1_X \otimes e"'] & X \arrow[l, "\rho_X^{-1}"'] \arrow[r, "\lambda_X^{-1}"] \arrow[d, "1_X"] & I \otimes X \arrow[d, "e \otimes 1_X"] \\
X \otimes X \arrow[r, "\bullet"'] & X & X \otimes X \arrow[l, "\bullet"]
\end{tikzcd}$$

(Exercise $I \xrightarrow{e} X$ is uniquely determined by $X \otimes X \xrightarrow{\bullet} X$.)

A homomorphism from $(X, \bullet, e)$ to $(X', \bullet', e')$ is a map $X \xrightarrow{h} X'$ s.t.

$$\begin{tikzcd}
X \otimes X \arrow[r, "h \otimes h"] \arrow[d, "\bullet"'] & X' \otimes X' \arrow[d, "\bullet'"] \\
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
X \arrow[rr, "h"] & & X'
\end{tikzcd}$$

$\mathrm{Mon}_{\mathcal{C}}$: The category of monoids in $\mathcal{C}$ and homomorphisms

Examples

A monoid in $\underline{\mathrm{Vect}}_K$ is a vector space V

with $V \otimes V \xrightarrow{\cdot} V$ and $K \xrightarrow{e} V$ corresponding to

a function $V \times V \xrightarrow{\cdot} V$ and element $\underline{1} \in V$

satisfying

bilinearity

$$k\underline{u} \cdot \underline{w} = k(\underline{u} \cdot \underline{w}) \qquad (\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$$

$$\underline{u} \cdot k\underline{w} = k(\underline{u} \cdot \underline{w}) \qquad \underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$$

monoid laws

$$\underline{u} \cdot (\underline{v} \cdot \underline{w}) = (\underline{u} \cdot \underline{v}) \cdot \underline{w}$$

$$\underline{u} \cdot \underline{1} = \underline{u} = \underline{1} \cdot \underline{u}$$

I.e. a monoid in $\underline{\mathrm{Vect}}_K$ is exactly an <u>associative K-algebra</u>

A monoid in <u>Cat</u> (w.r.t. product $\times$) is exactly a (small) strict monoidal category

It is often useful to define varieties of mathematical structure (e.g. algebraic structures) internally in the context of an ambient category C. What we can define in this way depends on the category-theoretic structure of C we use.

Monoidal structure : algebraic structures defined using equations in which the same variables appear in the same order on each side of the equation, and each variable appears only once on each side. (E.g. monoids: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $x \cdot e = x = e \cdot x$.)

Symmetric monoidal structure : as above, except that the variables are not required to appear in the same order on both sides of an equation. (E.g. commutative monoides. $x \cdot y = y \cdot x$.)

Finite products : general algebraic structures defined using equations (e.g. groups, Abelian groups, rings).

Finite limits : more exotic structures with, for example, partially defined operations whose domains have 'positive' descriptions (e.g. categories - composition is partially defined in the sense that composable arrows have to have matching domain/codomain).

Rings and modules in $\mathcal{C}$ with finite products.

A *ring* (with unit) is given by an object R and maps

$$1 \xrightarrow{0} R \qquad R \times R \xrightarrow{+} R \qquad 1 \xrightarrow{1} R \qquad R \times R \xrightarrow{\cdot} R$$

satisfying diagrams expressing the usual laws. Exercise work these out.

Example In Top the rings are exactly the topological rings.

Exercise Define the category $\mathrm{Ring}_{\mathcal{C}}$ of rings (with unit) and ring homomorphisms.

We can also formulate the derived algebraic notion of *R-module*

An R-module in $\mathcal{C}$ is $(X, \underline{0}, \underline{+}, \underline{\cdot})$ where

$$1 \xrightarrow{\underline{0}} X \qquad X \times X \xrightarrow{\underline{+}} X \qquad R \times X \xrightarrow{\underline{\cdot}} X$$

satisfy diagrams expressing the usual laws; e.g.

$$\begin{tikzcd}
R \times (R \times X) \arrow[r, "1_R \times \underline{\cdot}"] \arrow[d, "\alpha"'] & R \times X \arrow[dr, "\underline{\cdot}"] & \\
& & X \\
(R \times R) \times X \arrow[r, "\cdot \times 1_X"] & R \times X \arrow[ur, "\underline{\cdot}"'] &
\end{tikzcd}$$

$$\left(r_1 \underline{\cdot} (r_2 \underline{\cdot} x) = (r_1 \cdot r_2) \underline{\cdot} x \right)$$

Similarly a homomorphism of modules is ... (Exercise!)

If R is a field in Set then we obtain the category Vect_R

If R is $\mathbb{R}$ or $\mathbb{C}$ in Top then we obtain topological vector spaces etc.

Internal categories in a category C with finite limits

An *internal category* is a tuple $\mathbb{D} = (D_0, D_1, d_0, d_1, \underline{id}, \underline{\circ})$

where: $D_0, D_1 \in |C|$

D_0 ~ object of objects

D_1 ~ object of morphisms

$$\begin{tikzcd} D_0 \arrow[r, "\underline{id}"] & D_1 \arrow[l, bend right=60, "d_1"'] \arrow[l, bend left=60, "d_0"] \end{tikzcd}$$

are maps in C with $d_0 \circ \underline{id} = 1_{D_0} = d_1 \circ \underline{id}$

d_0 & d_1 ~ domain and codomain of a map

$\underline{id}$ ~ identity map on an object

$$D_1 \times_{D_0} D_1 \xrightarrow{\underline{\circ}} D_1$$

where

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \arrow[d, "d_1"] \\ D_1 \arrow[r, "d_0"'] & D_0 \end{tikzcd}$$

$\underline{\circ}$ ~ composition of maps

Such that

$$\begin{tikzcd} D_1 \times_{D_0} D_0 \arrow[d, "1_{D_1} \times_{D_0} \underline{id}"'] & D_1 \arrow[l, "\langle 1_{D_1}, d_0 \rangle"'] \arrow[r, "\langle d_1, 1_{D_1} \rangle"] \arrow[d, "1_{D_1}"] & D_0 \times_{D_0} D_1 \arrow[d, "\underline{id} \times_{D_0} 1_{D_1}"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\underline{\circ}"] & D_1 & D_1 \times_{D_0} D_1 \arrow[l, "\underline{\circ}"'] \end{tikzcd}$$

$\underline{id}$ gives right identities w.r.t $\underline{\circ}$

$\underline{id}$ gives left identities w.r.t. $\underline{\circ}$

and

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \times_{D_0} D_1 \arrow[r, "1_{D_1} \times_{D_0} \underline{\circ}"] \arrow[d, "\underline{\circ} \times_{D_0} 1_{D_1}"'] & D_1 \times_{D_0} D_1 \arrow[d, "\underline{\circ}"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\underline{\circ}"'] & D_1 \end{tikzcd}$$

composition is associative

The top left vertex is constructed as the pullback

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \times_{D_0} D_1 \arrow[r, "\langle \pi_2, \pi_3 \rangle"] \arrow[d, "\langle \pi_1, \pi_2 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \times_{D_0} D_1 \arrow[d, "\pi_1"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\pi_2"'] & D_1 \end{tikzcd}$$

The object of composable triples of morphisms.

(The notation $D_1 \times_{D_0} D_1 \times_{D_0} D_1$ is not very precise! Also my verbal description of this construction in the lecture was a bit misleading.)

Exercise

- Give precise constructions of the maps $\circ \times_{D_0} 1_{D_1}$ and $1_{D_1} \times_{D_0} \circ$ that appear in the associativity diagram.

- Define internal functor between internal categories.

- Define internal natural transformation between internal functors

Week 9 puzzle

(1) What is a monoid in the strict monoidal category $[\mathcal{C},\mathcal{C}]$ of endofunctors on a category $\mathcal{C}$?

I am looking for an answer of the form: a monoid is an endofunctor together with certain natural transformations satisfying certain properties.

(2) Find natural examples of such monoids.

This puzzle will be answered in the Week 10 lecture

Category Theory 2022-23

Lecture 10

9th December 2022

$\underline{Grp}$ is a typical category of algebras and their homomorphisms.

The forgetful functor $U : \underline{Grp} \to \underline{Set}$ has a left adjoint, the free group functor $F : \underline{Set} \to \underline{Grp}$

For any set X, group G and function $f : X \to G$ there is a unique homomorphism $h : FX \to G$ s.t.

$$\begin{tikzcd} FX \arrow[r, "h"] & G \\ X \arrow[u, "i_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

where $i_X(x)$ = the generator in FX associated to x.

The free group FX can be constructed explicitly as equivalence classes of certain expressions

This is an example of a general method for constructing free algebras explicitly

Such constructions are "fussy" Mac Lane [CWM]

CSL (Complete semilattices)

Another category of algebra-like structures

A *complete semilattice* is a partially ordered set $(X, \leq)$ in which every subset $A \subseteq X$ supremum (least upper bound) $\bigvee A \in X$.

Every complete semilattice has a least element ($\bigvee \emptyset$).

It also has all infima (greatest lower bounds) $\bigwedge A$

(I.e a complete semilattice is always a complete lattice.)

A homomorphism of CSLs is a function $h : X \to Y$ that preserves suprema

$$\forall A \subseteq X \quad h\left(\bigvee_X A\right) = \bigvee_{X'} h(A)$$

direct image

It follows that homomorphisms are order preserving

$$x \leq_X x' \Rightarrow h(x) \leq_{X'} h(x')$$

(Homorphisms preserve least element, but need not preserve $\bigwedge$.)

Alternative Formulation

A complete semilattice is a structure $(X, \bigvee)$ where $\bigvee : \mathcal{P}X \to X$ satisfies

$$\bigvee\{x\} = x \qquad \forall x \in X$$

$$\bigvee\{\bigvee B \mid B \in \mathcal{A}\} = \bigvee(\bigcup \mathcal{A}) \qquad \forall \mathcal{A} \subseteq \mathcal{P}\mathcal{B}$$

Prop For any set X, csl $(Y, \bigvee)$ and function $f : X \to Y$, there exists a unique csl homomorphism $h : (\mathcal{P}X, \bigcup) \longrightarrow (Y, \bigvee)$ s.t.

$$\begin{tikzcd} \mathcal{P}X \arrow[r, "h"] & Y \\ X \arrow[u, "\{\cdot\}"] \arrow[ur, "f"'] & \end{tikzcd}$$

Equivalently The left adjoint to the forgetful functor $U : \underline{CSL} \to \underline{Set}$ is

$$\begin{array}{c} X \\ f \downarrow \\ Y \end{array} \longmapsto \begin{array}{c} (\mathcal{P}X, \bigcup) \\ \downarrow \; A \mapsto f(A) \\ (\mathcal{P}Y, \bigcup) \end{array}$$

One of the two covariant powerset functors from Lec. 2.

KHaus Objects: compact Hausdorff spaces
Morphisms: Continuous functions

Surprising facts
- The forgetful U: KHaus $\to$ Set has a left adjoint
- KHaus is a category of algebras & homomorphisms.

Neither of these points are easy to show.

Today we address

- How to find adjoints without constructing them explicitly (using adjoint functor theorems).

- When an adjunction can be viewed as a free-algebra construction for a category of algebras and homomorphism.

We shall use Grp, CSL and KHaus as our running examples.

Rather than constructing the left adjoint directly we prove that it exists using Freyd's <u>adjoint functor theorem</u>

<u>FAFT</u> Suppose $G : \mathcal{D} \to \mathcal{C}$ is a functor from a complete category $\mathcal{D}$ and G preserves limits. Then G has a left adjoint if and only if it enjoys the following <u>solution set condition</u>: for any $X \in |\mathcal{C}|$, there exists a family $(X \xrightarrow{g_i} GY_i)_{i \in I}$ of maps in $\mathcal{C}$ indexed by a set I, such that, for every map $X \xrightarrow{f} GY$ in $\mathcal{C}$ there exist $i \in I$ and $Y_i \xrightarrow{h} Y$ in $\mathcal{D}$ such that $f = Gh \circ g_i$.

Recommended: • Read the proof

• Read about the Special Adjoint Functor Theorem

See [CWM] (or other textbook)

We prove that <u>KHaus</u> is complete and $U : \underline{\text{KHaus}} \to \underline{\text{Set}}$ preserves limits simultaneously by proving a stronger property: U creates limits.

A functor $U : \mathcal{A} \to \mathcal{C}$ <u>creates limits</u> if, for any diagram $D : G \to \mathcal{A}$ (G a graph), and any $\{L \xrightarrow{q_x} UD_x\}_{x \in G}$ limit cone in $\mathcal{C}$ for the diagram UD, we have:

- there exists a unique D-cone $\{K \xrightarrow{p_x} D_x\}_{x \in G}$ in $\mathcal{A}$ that is mapped by U to $\{L \xrightarrow{q_x} UD_x\}_{x \in G}$; and
- $\{K \xrightarrow{p_x} D_x\}_{x \in G}$ is a limit cone for D in $\mathcal{A}$.

(The above formulation is standard and useful. However, it breaks a category-theoretic taboo, it is not stable under equivalence of categories.)

$U : \underline{\mathrm{KHaus}} \to \underline{\mathrm{Set}}$ creates products (& equalisers)

Let $(X_i)_{i \in I}$ be a family of Comp. Haus. space.

By Tychonoff's theorem (requires AC!) the product topological space $\prod_{i \in I} X_i$ is itself a compact Hausdorff space.

Since $\left(\prod_{i \in I} X_i \xrightarrow{\pi_i} X_i\right)_{i \in I}$ is a cone in $\underline{\mathrm{KHaus}}$ that is also a product cone in $\underline{\mathrm{Top}}$ it is a fortiori a limit cone in $\underline{\mathrm{KHaus}}$.

Now let $(P \xrightarrow{p_i} UX_i)$ be any product cone in $\underline{\mathrm{Set}}$.

Topologise P with the unique topology that turns the product isomorphism $P \xrightarrow{i} \prod_{i \in I} X_i$ in $\underline{\mathrm{Set}}$ into a homeomorphism.

This is again compact Hausdorff since $\prod_{i \in I} X_i$ is.

Moreover, any other topology on P for which the cone $(P \xrightarrow{p_i} X_i)$ remains continuous would have to be finer than the given one since the product topology is the coarsest such topology.

But no strictly finer topology can be compact Hausdorff.

So U creates products. A similar argument shows it creates equalisers, hence arbitrary limits.

The solution set condition for $U : \underline{\mathrm{kHaus}} \to \underline{\mathrm{Set}}$

Given a set X, consider any compact Hausdorff space Y and function $f : X \to Y$ with dense image. Then the function $y \mapsto \{A \subseteq X \mid y \in \overline{f(A)}\}$ (where $\overline{f(A)}$ is the closure of $f(A)$ in Y) is injective. So Y is in bijective correspondence with a subset of $\wp\wp X$.

Consider the family $\left(g : X \to Z\right)_{(g, Z, \tau)}$

indexed by the set

$$\{(g, Z, \tau) \mid Z \subseteq \wp\wp X,\ \tau \text{ is a compact Hausdorff topology on } Z,\ g : X \to Z \text{ has dense image}\}$$

This satisfies the required condition. Consider any function $f : X \to Y$ where Y is a compact Haus. space.

Let $Y' := \overline{\mathrm{image}(f)}$. Since $Y' \subseteq Y$ is closed it carries a compact Hausdorff topology. Clearly $f_{Y'} : X \to Y'$ has dense image. By the remarks at the start, Y' is homeomorphic to some compact Hausdorff space (Z, τ) with $Z \subseteq \wp\wp X$ via $i : Z \to Y'$. The function $g := i^{-1} \circ f_{Y'}$ indexed by (g, Z, τ) and continuous function $i : Z \to Y' \subseteq Y$ provide the required factorisation of f. $\square$

Revisiting CSLs.

The endofunctor $\mathcal{P} : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{P}X \arrow[d, "\mathcal{P}f := A \mapsto f(A)"] \\ \mathcal{P}Y \end{tikzcd}$$

carries the structure of a monoid in $[\underline{\mathrm{Set}}, \underline{\mathrm{Set}}]$

$$\{\cdot\}_X := x \mapsto \{x\} : X \to \mathcal{P}X \qquad \{\} : 1_{\underline{\mathrm{Set}}} \Rightarrow \mathcal{P}$$

$$\bigcup_X := A \mapsto \bigcup A : \mathcal{P}\mathcal{P}X \to \mathcal{P}X \qquad \bigcup : \mathcal{P}^2 \Rightarrow \mathcal{P}$$

$$\begin{tikzcd} & \mathcal{P}X \arrow[dl, "\{\cdot\}_{\mathcal{P}X}"'] \arrow[d, "1_{\mathcal{P}X}"] \arrow[dr, "\mathcal{P}\{\cdot\}_X"] & \\ \mathcal{P}^2X \arrow[r, "\bigcup_X"'] & \mathcal{P}X & \mathcal{P}^2X \arrow[l, "\bigcup_X"] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{P} \arrow[dl, Rightarrow, "\{\cdot\}\mathcal{P}"'] \arrow[d, Rightarrow, "1_{\mathcal{P}}"] \arrow[dr, Rightarrow, "\mathcal{P}\{\cdot\}"] & \\ \mathcal{P}^2 \arrow[r, Rightarrow, "\bigcup"'] & \mathcal{P} & \mathcal{P}^2 \arrow[l, Rightarrow, "\bigcup"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{P}^3X \arrow[r, "\mathcal{P}\bigcup_X"] \arrow[d, "\bigcup_{\mathcal{P}X}"'] & \mathcal{P}^2X \arrow[d, "\bigcup_X"] \\ \mathcal{P}^2X \arrow[r, "\bigcup_X"'] & \mathcal{P}X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}^3 \arrow[r, Rightarrow, "\mathcal{P}\bigcup"] \arrow[d, Rightarrow, "\bigcup\mathcal{P}"'] & \mathcal{P}^2 \arrow[d, Rightarrow, "\bigcup"] \\ \mathcal{P}^2 \arrow[r, Rightarrow, "\bigcup"'] & \mathcal{P} \end{tikzcd}$$

The algebraic redefinition of csls has an elegant definition involving the above structure

A <u>csl</u> (X, V) is $PX \xrightarrow{V} X$ such that

$$\begin{tikzcd} & X \arrow[dl, "\{\}_X"'] \arrow[d, "1_X"] \\ PX \arrow[r, "V"'] & X \end{tikzcd} \qquad \begin{tikzcd} P^2X \arrow[r, "PV"] \arrow[d, "U"'] & PX \arrow[d, "V"] \\ PX \arrow[r, "V"'] & X \end{tikzcd}$$

A <u>homomorphism</u> of csls from (X, V) to (X', V') is $X \xrightarrow{h} X'$ s.t.

$$\begin{tikzcd} PX \arrow[r, "Ph"] \arrow[d, "V"'] & PX' \arrow[d, "V"] \\ X \arrow[r, "h"'] & X' \end{tikzcd}$$

Definition A **monad** on $\mathcal{C}$ is a monoid in the strict monoidal category $[\mathcal{C},\mathcal{C}]$.

I.e. A monad is (T, η, μ) the unit the multiplication

where $T : \mathcal{C} \to \mathcal{C}$ is a functor

$\eta : 1_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are nat transfs.

Such that

$$\begin{tikzcd} & T \arrow[dl, Rightarrow, "\eta T"'] \arrow[d, Rightarrow, "1_T"] \arrow[dr, Rightarrow, "T\eta"] & \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T & T^2 \arrow[l, Rightarrow, "\mu"] \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

An (Eilenberg-Moore) <u>algebra</u> for a monad T is (A, a) where $A \in |\mathcal{C}|$ and $TA \xrightarrow{a} A$ is such that

$$\begin{tikzcd} & A \arrow[dl, "\eta_A"'] \arrow[dr, "1_A"] & \\ TA \arrow[rr, "a"'] & & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

A <u>homomorphism</u> of algebras from (A, a) to (B, b) is $A \xrightarrow{h} B$ in $\mathcal{C}$ such that

$$\begin{tikzcd} TA \arrow[r, "Th"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

N.B. The definition of a monad implies that (TX, μ_X) is always an algebra. By naturality of μ, for any $X \xrightarrow{f} Y$, we have Tf is a homomorphism from (TX, μ_X) to (TY, μ_Y)

The definition of algebra implies that for any algebra (A, a) the map $TA \xrightarrow{a} A$ is a homomorphism from (TA, μ) to (TA, a).

The Eilenberg-Moore category of algebras

If (T, η, μ) is a monad on $\mathcal{C}$, we write $\mathcal{C}^T$ for the category where objects are algebras for the monad T and whose maps are homomorphisms.

Let $U : \mathcal{C}^T \to \mathcal{C}$ be the forgetful functor

Prop U has a left adjoint $F : \mathcal{C} \to \mathcal{C}^T$

$$F : X \mapsto (TX, \mu_X)$$

$$\begin{tikzcd} X \arrow[d, "f"'] & & (TX, \mu_X) \arrow[d, "Tf"] \\ Y & \mapsto & (TY, \mu_Y) \end{tikzcd}$$

Prop U creates limits.

Exercise Prove the 2 propositions.

Adjunctions give rise to monads

Prop Any adjunction $F \dashv G : D \to C$ gives rise to a monad (T, η, μ) on C defined by

$$T := GF$$

$\eta : 1_C \Rightarrow GF$ is the unit of the adjunction

$\mu := G\varepsilon F : GFGF \Rightarrow GF$

where $\varepsilon : FG \Rightarrow 1_D$ is the adjunction counit

Given an adjunction and associated monad as above define the comparison functor $K : D \to C^T$ by

$$K(Y) := (GY,\ GFGY \xrightarrow{G\varepsilon_Y} GY)$$

$$\begin{array}{ccc} Y & & GY \\ \downarrow g & \longmapsto & \downarrow Gg \\ Y' & & GY' \end{array}$$

In the case of Grp, CSL and KHaus the comparison functor is an isomorphism of categories.!

The notion of algebra for a monad encompasses the familiar examples (groups, rings, modules, csls) of algebras (for which free-algebras exist).

It also explicates in what sense KHaus is a category of algebras.

Beck's theorem gives a beautiful method for proving that the comparison functor is an isomorphism.

Beck's (monadicity theorem)

For an adjunction $F \dashv G : D \to C$ the following are equivalent.

- G creates absolute coequalisers.
- The comparison functor $K : D \to C^{GF}$ is an isomorphism of categories.

(One says that the functor G is monadic.)

There are many variants of this theorem (see [CWM]) Some establishing conditions for showing that K is an equivalence of categories.

An absolute coequaliser in $\mathcal{C}$ is a coequaliser diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$$

that satisfies: for all categories $\mathcal{E}$ and functors $F : \mathcal{C} \to \mathcal{E}$, it holds that F preserves the above coequaliser.

$G : \mathcal{D} \to \mathcal{C}$ <u>creates absolute coequalisers</u> if: for any $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{D}$ and absolute coequaliser $GX \underset{Gg}{\overset{Gf}{\rightrightarrows}} GY \xrightarrow{q} Q$ in $\mathcal{C}$, there exists a unique $Y \xrightarrow{r} Z$ in $\mathcal{D}$ s.t. $GZ = Q$ and $Gr = q$; furthermore this unique r is a coequaliser for $X \underset{g}{\overset{f}{\rightrightarrows}} Y$.

<u>Outline proof</u> that $U : \underline{\mathrm{Grp}} \to \underline{\mathrm{Set}}$ creates absolute coequalisers.

Let $G \underset{h}{\overset{g}{\rightrightarrows}} H$ be group homomorphisms such that

$$G \underset{h}{\overset{g}{\rightrightarrows}} H \xrightarrow{q} Q \quad \text{is an absolute coequaliser in } \underline{\mathrm{Set}}$$

Since absolute, the coequaliser is preserved by $X \mapsto X \times X : \underline{\mathrm{Set}} \to \underline{\mathrm{Set}}$

$$\begin{tikzcd}
G^2 \arrow[r, shift left, "g^2"] \arrow[r, shift right, "h^2"'] \arrow[d, "\bullet_G"'] & H^2 \arrow[r, "q^2"] \arrow[d, "\bullet_H"] & Q^2 \arrow[d, dashed, "\bullet_Q"] \\
G \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & H \arrow[r, "q"'] & Q
\end{tikzcd}$$

We thus obtain a map $Q^2 \xrightarrow{\bullet_Q} Q$ as in the diagram, giving Q a group structure in the unique way making q a homomorphism. $\square$

Category Theory 2022–23

Lecture 11

16th December 2022

In **Set** objects are determined up to isomorphism by their global points

$$X \cong Y \iff \mathbf{Set}(1, X) \cong \mathbf{Set}(1, Y)$$

The corresponding property does not hold in an arbitrary category with terminal object.

(Exercise: find counterexamples.)

In an arbitrary category $\mathcal{C}$ we need to consider generalised points of X

maps $Z \longrightarrow X$ where Z ranges over all of $|\mathcal{C}|$

Generalised points form a contravariant functor in Z

$$\begin{array}{ccc} Z & & \mathcal{C}(Z,X) \\ g \uparrow & \longmapsto & \downarrow - \circ g \\ Z' & & \mathcal{C}(Z',X) \end{array}$$

We assume $\mathcal{C}$ is locally small

We write $\underline{y}X := \mathcal{C}(-,X) : \mathcal{C}^{op} \to \underline{Set}$

for this functor, the representable functor from X

(sometimes representable presheaf)

Proposition 1 (Corollary of the Yoneda Lemma to follow)

Objects are determined by generalised points; i.e.,

$$\underline{y}X \cong \underline{y}Y \text{ in } [\mathcal{C}^{op}, \underline{Set}] \iff X \cong Y \text{ in } \mathcal{C}$$

$[\mathcal{C}^{op}, \underline{Set}]$ is well-defined as a locally small category with a class of objects if and only if $\mathcal{C}$ is small. In the case of a locally small $\mathcal{C}$, the collection of objects of $[\mathcal{C}^{op}, \underline{Set}]$ lies outside the realm of sets and classes. Nevertheless one can make sense of $\underline{y}X \cong \underline{y}Y$, which refers to the hom class between 2 objects.

The mapping from object X to representable $\underline{y}X$ is itself a covariant functor

we henceforth assume $\mathcal{C}$ is small

the Yoneda functor $\underline{y} : \mathcal{C} \to [\mathcal{C}^{op}, \underline{Set}]$

$$\begin{array}{ccc} X & & \underline{y}X \\ g\downarrow & \longmapsto & \Downarrow \\ Y & & \underline{y}Y \end{array} \quad \left(Z \xrightarrow{f} X \mapsto g \circ f \right)_Z$$

One can equivalently obtain the above functor from our original hom functor

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{Set}$$

using the exponential property of functor categories

$$\frac{\mathcal{C}^{op} \times \mathcal{C} \to \underline{Set}}{\mathcal{C} \to [\mathcal{C}^{op}, \underline{Set}]}$$

Proposition 2 (Corollary of the Yoneda lemma to follow)

The Yoneda functor is full and faithful

Application Let G be a group.

$\underline{y} : \underline{G} \to [\underline{G}^{op}, \underline{Set}]$ is (full and) faithful.

Since $\underline{G}$ has only one object,

$U := \alpha \mapsto \alpha_* : [\underline{G}^{op}, \underline{Set}] \to \underline{Set}$ is faithful.

So $U\underline{y} : \underline{G} \to \underline{Set}$ is faithful.

Since functors preserve isos this gives a faithful functor

$$U\underline{y} : \underline{G} \to \underline{Set}_{iso}$$

i.e., we have embedded G in a symmetric group, proving Cayley's theorem.

The argument works in ignorance!

One does not need to know that $[\underline{G}^{op}, \underline{Set}]$ is isomorphic to the category of right G actions (cf. week 3 puzzle).

One does not need to know that $\underline{y}*$ is the transitive right action of G on itself.

The Yoneda lemma

For any $F : \mathcal{C}^{op} \to \underline{\mathrm{Set}}$ and $X \in |\mathcal{C}|$

$$[\mathcal{C}^{op}, \underline{\mathrm{Set}}](\underset{\sim}{y}X, F) \cong FX$$

naturally in X and F.

<u>Proof idea</u>

The required bijections are

$$w \in FX \mapsto \left(Z \xrightarrow{f} X \mapsto F(f)(w) \right)_Z$$

$$\alpha : \underset{\sim}{y}X \Rightarrow F \mapsto \alpha_X(1_X)$$

One must then verify that these are mutual inverses and the naturality property. This is routine. $\square$

Proof of Prop 2: $\underline{y}$ is full & faithful

The Yoneda lemma gives

$$[C^{op}, \underline{Set}](\underline{y}X, \underline{y}Y) \cong \underline{y}Y(X) = C(X,Y)$$

where the right-to-left bijection is

$$X \xrightarrow{g} Y \mapsto (Z \xrightarrow{f} X \mapsto g \circ f)$$

which is the morphism action of $\underline{y}$ $\square$

Proof of Prop 1: $\underline{y}X \cong \underline{y}Y \iff X \cong Y$.

$\underline{y}$ is full and faithful It therefore

creates isos. $\square$

$F : C \to D$ <u>creates isos</u> if, for any $X, Y \in |C|$

and iso $g : FX \to FY$ in D, there exists a unique

$f : X \to Y$ s.t $Ff = g$; moreover, this unique f is an iso.

For a <u>small</u> category $\mathbb{C}$, the Yoneda Functor gives a full & faithful embedding of $\mathbb{C}$ into $[\mathbb{C}^{op}, \underline{Set}]$ its category of <u>presheaves</u>.

(other standard notation: $\hat{\mathbb{C}}$, $Psh(\mathbb{C})$)

$$\mathbb{C} \xhookrightarrow{\underline{y}} Psh(\mathbb{C})$$

preserves existing limits

(but <u>not</u> colimits)

complete, cocomplete, cartesian closed, a (Grothendieck) topos

Limits and colimits in $\mathrm{PSh}(\mathbb{C})$ are computed pointwise as in Set

E.g., given presheaves F and G on $\mathbb{C}$.

Define the product presheaf $F \times G$ by:

$$(F \times G)(X) := FX \times GX$$

product in Set

$$F \times G : \begin{array}{c} X \\ {\scriptstyle f}\big\uparrow \\ X' \end{array} \longmapsto \begin{array}{c} FX \times GX \\ \big\downarrow {\scriptstyle Ff \times Gf} \\ FX' \times GX' \end{array}$$

Theorem $\mathrm{Psh}(\mathbb{C})$ is cartesian closed.

Proof outline Given presheaves F, G we need to find an exponential presheaf $[F, G]$.

Suppose such a presheaf exists, then it must enjoy the following properties

$$[F,G](X) \cong \mathrm{Psh}(\mathbb{C})\left(\underline{y}X, [F,G]\right) \quad \text{(Yoneda)}$$

$$\cong \mathrm{Psh}(\mathbb{C})(\underline{y}X \times F, G) \quad \text{(defining property of } [F,G])$$

Accordingly, we define

$$[F,G](X) := \mathrm{Psh}(\mathbb{C})(\underline{y}X \times F, G)$$

The corresponding morphism action is determined by the naturality of the above bijections.

Explicitly the morphism action is

A

$$\begin{tikzcd} X & & \hat{\mathbb{C}}(\underset{\sim}{y}X \times F, G) \arrow[d, "\alpha \mapsto \left((g,w) \in \mathbb{C}(z,Y) \times F(z) \mapsto \alpha_z(f \circ g, y)\right)_z"] \\ Y \arrow[u, "f"] & \longmapsto & \hat{\mathbb{C}}(\underset{\sim}{y}Y \times F, G) \end{tikzcd}$$

One now needs to check that the presheaf so defined indeed satisfies the properties required of $[F, G]$.

This is left as an exercise (for the enthusiastic only)

Example $\underline{Psh}(G)$ (i.e. $[G^{op}, \underline{Set}]$ cf. week 7 puzzle)

$\underline{Psh}(G) \cong$ category of right G-actions (week 3 puzzle)

Let $\underline{A}, \underline{B}$ be presheaves corresponding to right G-actions $(A, \bullet_A)$, $(B, \bullet_B)$

We calculate $[\underline{A}, \underline{B}]$ in $\underline{Psh}(G)$.

since $\times$ is symmetric it doesn't matter that we have swapped the order

$$[\underline{A}, \underline{B}](*) \cong \underline{Psh}(G)(\underline{A} \times \underline{y}(*), \underline{B})$$

$$\cong \underline{G\text{-Act}_R}(\underline{A} \times G, \underline{B})$$

G with its self-right-action

$$\cong B^A \qquad \text{(set of } \underline{\text{all}} \text{ functions } A \to B\text{)}.$$

For the last bijection, any function $f : A \to B$ determines $\tilde{f} : A \times G \to B$ by

$$\tilde{f}(a, g) := f(a \bullet_A g^{-1}) \bullet_B g$$

We show that $\tilde{f}$ is equivariant

$$\tilde{f}((a,g)\cdot h) = \tilde{f}(a\cdot h, g\cdot h)$$
$$= f(a\cdot h\cdot h^{-1}\cdot g^{-1})\cdot g\cdot h$$
$$= f(a\cdot g^{-1})\cdot g\cdot h \quad = \quad \tilde{f}(a,g)\cdot h$$

The mapping $f \mapsto \tilde{f}$ gives the required bijection from B^A to $\underline{G\text{-Act}}_R(A\times G, B)$, with inverse $\phi \mapsto (a \mapsto \phi(a,e))$ for equivariant ϕ (e the group identity)

The presheaf structure on $[\underline{A}, \underline{B}]$ (see A) corresponds to the following right action on $\underline{G\text{-Act}}_R(A\times G, B)$

For equivariant $\phi : A\times G \to B$ and $g\in G$

$$\phi\cdot g : (a,h) \mapsto \phi(a, h\cdot g) \qquad \text{B}$$

Via the bijection $\psi : f \mapsto \tilde{f}$, we obtain the following isomorphic action on B^A

$$(f\cdot g)(a) = (\psi^{-1}(\psi(f)\cdot g))(a) = (\tilde{f}\cdot g)(a,e)$$
$$\overset{\text{B}}{=} \tilde{f}(a,g) = f(a\cdot g^{-1})\cdot g.$$

This explicitly defines the exponential $[A, B]$ in $\underline{G\text{-Act}}_R$.

Category Theory 2022-23

Lecture 12

23rd December 2022

The simplicial category $\underline{\Delta}$

Objects: sets $[n] := \{0, \dots, n\}$ $n \geq 0$

(N.B. $[n]$ has $n+1$ elements)

Morphisms from $[m]$ to $[n]$: order preserving functions $f: [m] \to [n]$

(i.e. $i \leq j \Rightarrow f(i) \leq f(j)$)

A simplicial set is a presheaf $\underline{\Delta}^{op} \to \underline{Set}$

$\underline{sSet} := Psh(\underline{\Delta})$ The category of simplicial sets

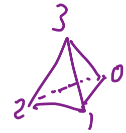
A mathematically important example of a presheaf category

Representable simplicial sets Δ^n = the standard n-simplex

$$\Delta^n := \underset{\sim}{y}[n] = \underline{\Delta}(-,[n])$$

We write Δ^n_m for $\Delta^n[m] = \underline{\Delta}([m],[n])$.

$\Delta^n_m \sim$ the m-simplices within the n-simplex

e.g. the 3-simplex is the tetrahedron 

Δ^3_0 has 4 elements (4 vertices)

Δ^3_1 has 6 non-degenerate elements (6 edges)
(non-degenerate = injective function)
and 10 elements (6 edges + 4 vertices)

Δ^3_2 has 4 non-degenerate elements (4 triangles)

Δ^3_3 has 1 non-degenerate element (1 tetrahedron)

Δ^3_m has 0 non-degenerate elements if $m > 3$.

A functor $\Delta : \underline{\Delta} \to \underline{Top}$

$$[n] \mapsto \Delta_n := \left\{ (x_0, \dots, x_n) \in [0,1]^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}$$

the standard topological n-simplex

$$\begin{array}{ccc} [m] & & \Delta_m \\ f\downarrow & \mapsto & \downarrow \\ [n] & & \Delta_n \end{array}$$

the unique affine function mapping $\hat{i}$ to $\widehat{f(i)}$ for $i \in [m]$

$\hat{i} = (x_0, \dots, x_n)$ where $x_j = \delta_{ij}$

Kronecker delta

A functor $S : \underline{Top} \to \underline{sSet}$

$$T \mapsto \underline{Top}(\Delta -, T)$$

$S(T)[n]$ = all continuous maps from Δ_n to T

$S(T)$ is the <u>total singular complex of T</u>

Any simplicial set encodes a topological space obtained intuitively by gluing together standard top. simplices according to the recipe encoded by the simplicial set

The <u>geometric realisation</u> functor $G : \underline{sSet} \to \underline{Top}$ is:

- the unique (up to natural isomorphism) colimit preserving functor such that

$$\begin{tikzcd} \underline{sSet} \arrow[r, "G"] & \underline{Top} \\ \underline{\Delta} \arrow[u, "\underline{y}"] \arrow[ur, "\Delta"'] \arrow[ur, phantom, near start, "\cong"] & \end{tikzcd}$$

- constructed explicitly as a <u>pointwise left Kan extension</u> of Δ along $\underline{y}$

- left adjoint to S: $\quad G \dashv S : \underline{Top} \to \underline{sSet}$

The first property above captures the gluing intuition.

The category of elements of a presheaf

Let $P : \mathbb{C}^{op} \to \underline{Set}$ be a presheaf.

The category $\int P$ of elements has

<u>Objects</u> (X, x) $\quad X \in |\mathbb{C}|, \quad x \in PX$

<u>Morphisms</u> from (X, x) to (Y, y):

maps $X \xrightarrow{f} Y$ in $\mathbb{C}$ s.t. $x = P(f)(y)$

($\equiv$ly s.t. $x = y \cdot f$ using 'action notation' for presheaves)

There is an obvious forgetful functor
$U : \int P \to \mathbb{C}$

Theorem (The co-Yoneda lemma!)

Every presheaf P is a colimit in $\mathrm{Psh}(\mathbb{C})$ of the diagram $\int P \xrightarrow{u} \mathbb{C} \xrightarrow{\underline{y}} \mathrm{Psh}(\mathbb{C})$.

This is often paraphrased: every presheaf is a colimit of representables. It is frequently referred to as "the co-Yoneda lemma", but it is not the only result known by this name.

Proof outline

The colimiting cocone $\left(\underline{y}X \xrightarrow{c_{(X,x)}} P\right)_{(X,x)}$

is given by

$$c_{(X,x)} := \Psi_X(x) \quad \text{where} \quad \Psi_X^P : P(X) \xrightarrow{\cong} \mathrm{Psh}(\mathbb{C})(\underline{y}X, P)$$

is the Yoneda lemma bijection

Given any other cocone $\left(\underline{y}X \xrightarrow{d_{(X,x)}} Q \right)_{(X,x)}$

we need to define the unique cocone morphism

$P \xrightarrow{e} Q$ in $\mathrm{Psh}(\mathbb{C})$.

The component $e_X : PX \to QX$

is the function

$$x \in PX \longmapsto \left(\psi^Q_X\right)^{-1}\left(d_{(X,x)}\right).$$

One then needs to verify.

- $\left(c_{(X,x)}\right)_{(X,x)}$ is indeed a cocone
- $(e_X)_X$ is natural
- e is a morphism of cocones
- e is the unique cocone morphism

$\square$

An alternative description of $\int P$.

<u>Objects</u> (X, α) where $X \in |\mathbb{C}|$ and $\underline{y}X \xrightarrow{\alpha} P$ in $\mathrm{Psh}(\mathbb{C})$

<u>Morphisms</u> from (X,α) to (Y,β) are maps $X \xrightarrow{f} Y$ in $\mathbb{C}$ such that

$$\begin{tikzcd} \underline{y}X \arrow[r, "\alpha"] \arrow[d, "\underline{y}f"'] & P \\ \underline{y}Y \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes in $\mathrm{Psh}(\mathbb{C})$.

This is an isomorphic category to $\int P$ by the Yoneda lemma.

It is an instance of a general <u>comma category</u> construction.

Given $F : C \to \mathbb{D}$ and $Z \in |\mathbb{D}|$ the <u>comma category</u> $F \downarrow Z$ has:

<u>Objects</u> (X, g) where $X \in |C|$ and $FX \xrightarrow{g} Z$ in $\mathbb{D}$

<u>Morphisms</u> from (X,g) to (Y,h) are maps $X \xrightarrow{f} Y$ in C s.t. $h \circ Ff = g$ in $\mathbb{D}$.

(More general comma category constructions than this exist too.)

The reformulation of $\int P$ at the top of the page shows that $\int P$ is isomorphic to the comma category $\underline{y} \downarrow P$.

Given $F : \mathbb{C} \to \mathbb{D}$ and $Z \in |\mathbb{D}|$ we have a diagram

$$F \downarrow Z \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathbb{D} \quad \text{in } \mathbb{D} \qquad (A)$$

which has a canonical cocone with vertex Z

$$\left(FX \xrightarrow{g} Z \right)_{(X,g)} \qquad (B)$$

A functor F is said to be <u>dense</u> if, for every $Z \in |\mathbb{D}|$, (B) is a colimit of (A).

<u>Reformulation</u> (the co-Yoneda lemma)

The Yoneda functor $\underline{y} : \mathbb{C} \to \mathrm{Psh}(\mathbb{C})$ is dense.

Theorem ($\mathrm{Psh}(\mathbb{C})$ is the free cocompletion of $\mathbb{C}$)

Let $\mathbb{C}$ be a small category.

For any cocomplete category $\mathcal{A}$ and functor $F : \mathbb{C} \to \mathcal{A}$, there exists a colimit preserving funtor $\overline{F} : \mathbb{C} \to \mathcal{A}$ such that

- $\overline{F}\,\underline{y} \cong F$ in $[\mathbb{C}, \mathcal{A}]$
- For any colimit preserving $\overline{G} : \mathrm{Psh}(\mathbb{C}) \to \mathcal{A}$ for which $\overline{G}\,\underline{y} \cong F$, it holds that $\overline{G} \cong \overline{F}$

$$\begin{tikzcd}
\mathrm{Psh}(\mathbb{C}) \arrow[r, "\overline{F}/\overline{G}", "\text{colimit preserving}"'] & \mathcal{A} \\
\mathbb{C} \arrow[u, hook, "\underline{y}"] \arrow[ur, "F"'] \arrow[ur, phantom, "\cong", near start, shift left=3]
\end{tikzcd}$$

More briefly, for any cocomplete $\mathcal{A}$ and functor $F : \mathbb{C} \to \mathcal{A}$, there exists a unique (up to natural isomorphism) colimit preserving functor $\overline{F} : \mathrm{Psh}(\mathbb{C}) \to \mathcal{A}$ such that $\overline{F}\,\underline{y} \cong F$.

The above property characterises $\mathrm{Psh}(\mathbb{C})$ up to equivalence of categories.

Given functors $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{A}$ ($\mathcal{A}, \mathcal{C}, \mathcal{D}$ arbitrary categories),

a <u>left Kan extension</u> of F along K ($\mathrm{Lan}_K F$)

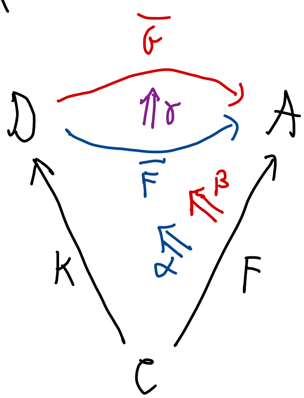
is a functor $\underline{F} : \mathcal{D} \to \mathcal{A}$ and natural transformation

$\alpha : F \Rightarrow \underline{F} K$ such that, for any $\underline{G} : \mathcal{D} \to \mathcal{A}$ and

natural transformation $\beta : F \Rightarrow \underline{G} K$, there exists

a unique natural transformation $\gamma : \underline{F} \Rightarrow \underline{G}$ such that

$$\beta = (\gamma K) \circ \alpha$$



<u>Proposition</u> Left Kan extensions (if they exist) are uniquely determined up to natural isomorphism.

Theorem Suppose $\mathbb{C}$ is small, $\mathbb{D}$ locally small and $\mathcal{A}$ cocomplete. Then every $F : \mathbb{C} \to \mathcal{A}$ has a left Kan extension along every $K : \mathbb{C} \to \mathbb{D}$, given explicitly by

$$(\mathrm{Lan}_K F)\, Y := \varinjlim \left(K \downarrow Y \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathcal{A} \right)$$

$\varinjlim$ means colimit, $\varprojlim$ means limit

With the morphism action determined by the universal property of colimits

If a left Kan extension is defined in the above way it is said to be a <u>pointwise</u> left Kan extension.

Proposition If $K : C \to D$ is full and faithful, and $(\bar{F} : D \to A,\ \alpha : F \Rightarrow \bar{F}K)$ is a pointwise left Kan extension of F along K then α is a natural isomorphism.

$$\begin{tikzcd} D \arrow[r, "\bar{F}"] & A \\ C \arrow[u, "K"] \arrow[ur, "F"'] \arrow[u, phantom, shift left=6, "\cong"] & \end{tikzcd}$$

I.e., $\bar{F}$ really does 'extend' F up to isomorphism

Proposition If $\mathcal{A}$ is cocomplete then for any $F : \mathbb{C} \to \mathcal{A}$ the functor $\mathrm{Lan}_{\underline{y}} F$: $\mathrm{Psh}(\mathbb{C}) \to \mathcal{A}$ is left adjoint to the functor $G : \mathcal{A} \to \mathrm{Psh}(\mathbb{C})$

$\bar{F}$ below

$$Z \mapsto \mathcal{A}(F-, Z) : \mathcal{A} \to \mathrm{Psh}(\mathbb{C})$$

Proof outline

$$\mathcal{A}(\bar{F}P, Z) \cong \mathcal{A}\left(\varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{F} \mathcal{A}\right), Z\right) \qquad \text{def. } \bar{F}$$

$$\cong \varprojlim\left(\left(\underline{y}\downarrow P\right)^{op} \xrightarrow{u} \mathbb{C}^{op} \xrightarrow{\mathcal{A}(F-, Z)} \underline{\mathrm{Set}}\right) \qquad \text{def } \varinjlim \; ⊛$$

$$\cong \varprojlim\left(\left(\underline{y}\downarrow P\right)^{op} \xrightarrow{u} \mathbb{C}^{op} \xrightarrow{GZ} \underline{\mathrm{Set}}\right) \qquad \text{def. } G$$

$$\cong \varprojlim\left(\left(\underline{y}\downarrow P\right)^{op} \xrightarrow{u} \mathbb{C}^{op} \xrightarrow{\mathrm{Psh}(\mathbb{C})(\underline{y}-, GZ)} \underline{\mathrm{Set}}\right) \qquad \text{Yoneda lemma}$$

$$\cong \mathrm{Psh}\left(\varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{\underline{y}} \mathrm{Psh}(\mathbb{C})\right), GZ\right) \qquad \text{def. } \varinjlim \; ⊛$$

$$\cong \mathrm{Psh}(P, GZ) \qquad \text{density of } \underline{y}.$$

□

⊛ Using $\mathbb{C}\left(\varinjlim\left(\mathbb{G} \xrightarrow{D} \mathbb{C}\right), Z\right) \cong \varprojlim\left(\mathbb{G}^{op} \xrightarrow{D} \mathbb{C}^{op} \xrightarrow{\mathbb{C}(-, Z)} \underline{\mathrm{Set}}\right)$

Proof that $Psh(\mathbb{C})$ is the free cocompletion of $\mathbb{C}$

Consider any functor $F : \mathbb{C} \to \mathcal{A}$ where $\mathcal{A}$ is cocomplete.

Define $\overline{F} : Psh(\mathbb{C}) \to \mathcal{A}$ to be the pointwise left Kan extension of F along $\underline{y}$, as given by the previous theorem.

$\overline{F}$ has a right adjoint $z \mapsto \mathcal{A}(F-, z)$, so $\overline{F}$ preserves colimits.

As a pointwise left k. ext. along a full & faithful functor ($\underline{y}$) we have $\overline{F}\underline{y} \cong F$.

Suppose $\overline{G}$ is another colimit preserving functor with $\overline{G}\underline{y} \cong F$.

Then.

$$\overline{G}P \cong \overline{G}\left(\varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{\underline{y}} Psh(\mathbb{C})\right)\right) \qquad \text{density of } \underline{y}$$

$$\cong \varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{\underline{y}} Psh(\mathbb{C}) \xrightarrow{\overline{G}} A\right) \qquad \overline{G} \text{ preserves colimits}$$

$$\cong \varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{F} A\right) \qquad \overline{G}\underline{y} \cong F$$

$$\cong \overline{F}P \qquad \text{definition of } \overline{F} \text{ as pointwise left Kan extension}$$

Where every isomorphism is natural in P.

$\square$

The nerve of a category

The *nerve* $N(\mathbb{C})$ of a small category $\mathbb{C}$ is the simplicial set

$$[n] \longmapsto \text{composable sequences } X_0 \xrightarrow{f_0} X_1 \to \cdots \xrightarrow{f_{n-1}} X_n$$

$$\begin{array}{ccc} [m] & & X_0 \xrightarrow{g_1} X_2 \to \cdots \xrightarrow{g_{m-1}} X_m \\ f \uparrow & \longmapsto & \downarrow \\ [n] & & X_{f(0)} \to X_{f(1)} \to \cdots \to X_{f(n)} \end{array}$$

where each $X_{f(i)} \longrightarrow X_{f(i+1)}$ is the composite $g_{f(i+1)-1} \circ \cdots \circ g_{f(i)}$ if $f(i+1) > f(i)$ and $1_{X_{f(i)}}$ if $f(i+1) = f(i)$.

The nerve functor $\quad N : \underline{\mathrm{Cat}} \to \underline{\mathrm{sSet}}$

Define $\underline{k} : \underline{\Delta} \to \underline{\mathrm{Cat}}$

$$\underline{k}[n] := \underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{\substack{n+1 \text{ objects} \\ n \text{ morphisms}}} = \underline{[n]}$$

(the partial order $([n], \leq)$ as a category)

$N : \underline{\mathrm{Cat}} \to \underline{\mathrm{sSet}}$ (cf. $\Delta : \underline{\mathrm{Top}} \to \underline{\mathrm{sSet}}$)

$$N(\mathbb{C}) = \underline{\mathrm{Cat}}(\underline{k}-, \mathbb{C})$$

As before, N has a left adjoint given by the unique (up to isomorphism) colimit preserving functor s.t.

$$\begin{tikzcd} \underline{\mathrm{sSet}} \arrow[r] & \underline{\mathrm{Cat}} \\ \underline{\Delta} \arrow[u, "\underline{y}"] \arrow[ur, "\underline{k}"'] & \end{tikzcd}$$

A new phenomenon:

The nerve functor $N: \underline{Cat} \to \underline{sSet}$ is full and faithful

Thus (small) categories can be viewed as special simplicial sets.

This viewpoint leads to the notion of <u>quasicategory</u> (a.k.a. <u>$(\infty,1)$-category</u>) which generalises ordinary categories to higher-dimensional 'categories' wrapped up as special simplicial sets.

Such quasicategories are the basis, for example, of Lurie's <u>Higher topos theory</u>

This and related approaches to higher-dimensional categories are a very active research area.

Category Theory 2022-23

Lecture 13

6th January 2023

Let S be a topological space

and $\mathcal{O}(S)$ the collection of open subsets of S

partially ordered by subset inclusion $(\subseteq)$

For $U \in \mathcal{O}(S)$ define

$$C(U) := \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$$

If $V \subseteq U$ then we have

$$f \longmapsto f\restriction_V \ : C(U) \to C(V)$$

The above defines a functor

$$C : \mathcal{O}(S)^{op} \longrightarrow \underline{\mathrm{Set}}$$ ($\mathcal{O}(S)$ qua category)

i.e., a presheaf.

The presheaf C satisfies a further condition expressing that the continuity of a function $f : U \to \mathbb{R}$ is determined locally within U

The sheaf property for C

Suppose $(U_i)_{i \in I}$ is an open cover of $U \in \mathcal{O}(S)$.

Suppose $(f_i : U_i \to \mathbb{R})_{i \in I}$ is a family of continuous functions such that

$$\forall i, j \in I \quad f_i \restriction_{U_i \cap U_j} = f_j \restriction_{U_i \cap U_j} : U_i \cap U_j \to \mathbb{R}$$

then there exists a unique continuous function

$$f : U \to \mathbb{R}$$

such that $\quad f \restriction_{U_i} = f_i \qquad \forall i \in I$.

Sheaf for a topological space

Let $P : \mathcal{O}(S)^{op} \to \underline{\mathrm{Set}}$ be a presheaf for S

A family $(x_i \in P(U_i))_{i \in I}$ is said to be

matching if

$$\forall i,j \in I \qquad x_i\restriction_{U_i \cap U_j} = x_j\restriction_{U_i \cap U_j} \in P(U_i \cap U_j)$$

(Given $x \in P(U)$ and $V \subseteq U$ we write $x\restriction_V$ for the element $P(\underbrace{U \to V}_{\text{the unique map } U \to V \text{ in } \mathcal{O}(S)})(x) \in P(V)$.)

An element $x \in P(U)$ where $U := \bigcup_{i \in I} U_i$ is

an amalgamation of $(x_i \in P(U_i))_{i \in I}$ if

$$\forall i \in I \qquad x\restriction_{U_i} = x_i$$

It is easy to show that any family $(x_i \in P(U_i))_{i \in I}$ that has an amalgamation is necessarily matching. (Exercise.)

<u>Definition</u> (Sheaf)

A presheaf is said to be a <u>sheaf</u> if every matching family has a unique amalgamation.

By our initial discussion

$$C(U) := \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$$

is a sheaf.

Other example sheaves on S

For any set X

$$F_X(U) := \{ f : U \to X \mid f \text{ any set-theoretic function} \}$$

For any topological space T

$$C_T(U) := \{ f : U \to T \mid f \text{ continuous} \}$$

If S, T are smooth manifolds

$$D_T(U) := \{ f : U \to T \mid f \text{ smooth} \}$$

If S, T are complex manifolds

$$H_T(U) := \{ f : U \to T \mid f \text{ holomorphic} \}$$

Let's fix $S, T := \mathbb{C}$ and look in more detail at the sheaf $H : \mathcal{O}(\mathbb{C})^{op} \to \underline{\text{Set}}$

$$H(U) := \{ h : U \to \mathbb{C} \mid h \text{ holomorphic} \}$$

$H(U)$ is indeed a presheaf

if $h : U \to \mathbb{C}$ is holomorphic and $V \subseteq U$ then $h\restriction_V : V \to \mathbb{C}$ is holomorphic

and moreover a sheaf

if $(h_i : U_i \to \mathbb{C})_{i \in I}$ is a matching family of holomorphic functions then the unique amalgamating function $h : U \to \mathbb{C}$ $(U := \bigcup_{i \in I} U_i)$ is holomorphic.

It further satisfies a more specific property

Given $h : U \to \mathbb{C}$ holomorphic and <u>connected</u> open $U' \supseteq U$ there is at most one holomorphic $h' : U' \to \mathbb{C}$ s.t. $h'\restriction_U = h$.

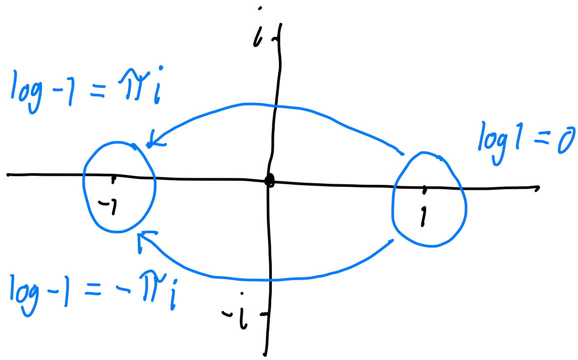
The last property suggests the idea of <u>analytic continuation</u>.

<u>Naively</u>:

Given holomorphic $h : U \to \mathbb{C}$ find as large as possible connected open $U' \supseteq U$ for which there exists a necessarily unique holomorphic $h' : U' \to \mathbb{C}$ with $h'\restriction_U = h$.

This is <u>too</u> naive!

E.g., we cannot extend the complex logarithm function in the above naive way



The full analytic continuation of $\log z$ does not define it on $\mathbb{C} - \{0\}$ but rather on an ascending/descending spiral surface above $\mathbb{C} - \{0\}$

the complex logarithm function

$$\begin{tikzcd}
\mathbb{R} \times \mathbb{R}_+ \arrow[rr, "(\theta, r) \mapsto \ln r + i\theta"] \arrow[dr, "(\theta, r) \mapsto re^{i\theta}"'] & & \mathbb{C} \arrow[dl, "z \mapsto e^z"] \\
& \mathbb{C} - \{0\} \arrow[d] & \\
& \mathbb{C} &
\end{tikzcd}$$

the domain of the full complex logarithm function is a <u>Riemann surface</u> over $\mathbb{C}$

Analytic continuations can be gathered together into a single Riemann surface, the <u>universal analytic function</u>

A <u>germ</u> is a pair $g = (z_0, (a_n)_{n\geq 0})$ such that the power series

$$F_g(z) := a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

Converges on an open subset of $\mathbb{C}$ containing z_0.

Let G be the set of germs with the following topology.
$V \subseteq G$ is <u>open</u> if:

for every $g = (z_0, (a_n)_{n\geq 0}) \in V$
there exists open $U_0 \subseteq \mathbb{C}$ with $z_0 \in U_0$ such that

- for every $z_0' \in U_0$, $F_g(z_0')$ is defined, <u>and</u>
- $g' \in V$ where g' is the unique germ $(z_0', (a_n')_{n\geq 0})$ satisfying

$$F_{g'}(z) = F_g(z) \text{ on some open neighbourhood of } z_0'$$

The above defines a topological space $\mathcal{G}$ together with a projection function

$$p := (z_0, (a_n)_{n\geq 0}) \mapsto z_0 \quad : \mathcal{G} \to \mathbb{C}$$

The function p is continuous. ($\mathcal{G}$ is a <u>bundle</u> over $\mathbb{C}$.)

The function p is also <u>étale</u>:

For every $g \in \mathcal{G}$ there exists open $V \subseteq \mathcal{G}$ with $g \in V$ such that $p(V) \subseteq \mathbb{C}$ is open and $p|_V : V \to p(V)$ is a homeomorphism

The space $\mathcal{G}$ is Hausdorff. (Exercise.)

The space $\mathcal{G}$ is known as the universal holomorphic function as it comprises <u>all</u> analytic functions.

It allows a precise definition of <u>analytic continuation</u>

Analytic continuations

Given a holomorphic function $h: U \to \mathbb{C}$ (for open $U \subseteq \mathbb{C}$) and $z_0 \in U$, let

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

be the power series expansion of h at z_0. Then the <u>analytic continuation</u> of h at z_0 is given by the connected component of $\mathcal{G}$ containing $(z_0, (a_n)_n)$.

The connected component of $\mathcal{G}$ containing $(z_0, (a_n)_n)$

$$\begin{tikzcd}
H \arrow[r, "g \mapsto F_g(p(g))"] \arrow[d, "p|_H"'] & \mathbb{C} \\
\mathbb{C} &
\end{tikzcd}$$

We have 2 seemingly very different mathematical structures embodying all holomorphic functions

- The sheaf $H : \mathcal{O}(\mathbb{C})^{op} \longrightarrow \underline{Set}$

  Models holomorphic functions <u>locally</u> $(U \to \mathbb{C})$

- The étale bundle $p : G \longrightarrow \mathbb{C}$

  Models holomorphic functions <u>globally</u>

Category theory shows that these two views of holomorphic functions arise as just one instance of a deep equivalence between sheaves and étale bundles valid for any topological space S.

The category $Sh(S)$ of <u>sheaves on S</u> is the full subcategory of $Psh(\mathcal{O}(S))$ on sheaves.

The category of <u>bundles over S</u> is just the slice category $\underline{Top}/S$.

Define $\Gamma : \underline{Top}/S \to Sh(S)$ by

$$\Gamma\left(\begin{array}{c} T \\ \downarrow p \\ S \end{array}\right)(U) = \left\{ s : U \to T \;\middle|\; \begin{array}{l} s \text{ is continuous and} \\ p(s(x)) = x \quad \forall x \in U \end{array} \right\}$$

$\Gamma(p)$ is the <u>sheaf of local sections</u> of p.

<u>Exercise</u> • Verify that $\Gamma(p)$ is indeed a sheaf.

• Define the morphism action of Γ.

A bundle $T \xrightarrow{p} S$ is <u>étale</u> if, for any $y \in T$, there exists open $V \subseteq T$ with $y \in V$ such that $p(V)$ is an open subset of S and $p : V \to p(V)$ is a homeomorphism.

(Étale maps are also known as local homeomorphisms.)

We write $\text{Étale}(S)$ for the full subcategory of $\underline{\mathrm{Top}}/S$ whose objects are étale maps.

<u>Theorem</u>

The functor $\Gamma : \text{Étale}(S) \to Sh(S)$ is (part of) an equivalence of categories.

Over a topological space S, sheaves are equivalent to étale bundles.

Outline proof

We define $\Lambda : Sh(S) \to \acute{E}tale(S)$,

which is the other half of the equivalence

Λ maps a sheaf F to its <u>bundle of germs</u>

Let $F : \mathcal{O}(S)^{op} \to \underline{Set}$ be a sheaf.

For any $x \in S$ consider the set

$$\{ (U,f) \mid U \in \mathcal{O}(S),\ x \in U,\ f \in F(U) \}$$

of elements of F local to x, with the equivalence relation

$$(U,f) \sim (U',f') \iff \exists U'' \subseteq U \cap U' \text{ s.t. } x \in U'' \text{ and } f\restriction_{U''} = f'\restriction_{U''}.$$

Given (U,f) as above the <u>germ of f at x</u> ($germ_x f$) is the equivalence class $[(U,f)]_\sim$.

We define a bundle $T_F \xrightarrow{p_F} S$

T_F has underlying set

$$T_F = \{(x, \mathrm{germ}_x f) \mid U \in \mathcal{O}(S), x \in U, f \in FU\}$$

A subset $V \subseteq T_F$ is <u>open</u> if it satisfies:

if $(x, \mathrm{germ}_x f) \in V$ where $f \in FU$

then $\exists U' \subseteq U$ with $x \in U'$ s.t.,

$(x', \mathrm{germ}_{x'}(f|_{U'})) \in V \quad \forall x' \in U'$.

(This topology is not in general Hausdorff, even when S is Hausdorff.)

The function $p_F : T_F \to S$ is first projection.

One verifies that p_F is continuous and étale.

The Functor $\Lambda : Sh(S) \to \text{Étale}(S)$

has as its action on objects

$$F \longmapsto \begin{array}{c} T_F \\ \big\downarrow {\scriptstyle p_F} \\ S \end{array}$$

One needs to further define the action on morphisms.

To show we have an equivalence of categories one finds natural isomorphisms

$$\begin{tikzcd} T_{\Gamma p} \arrow[rr, "\cong"] \arrow[dr, "p_{\Gamma p}"'] & \arrow[d, phantom, "A"] & T \arrow[dl, "p"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\cong", "B"'] & \Gamma \Lambda F \end{tikzcd}$$

A is where the property that p is étale is used

B is where the amalgamation property of F is used.

Category Theory 2022-23

Lecture 14

13th January 2023

A category $\mathcal{E}$ is an <u>elementary topos</u> if:

- it has finite limits
- is cartesian closed
- and has a <u>subobject classifier</u>:

an object Ω and map $1 \xrightarrow{\top} \Omega$ such that, for any mono $X \xrightarrow{m} Y$, there exists a unique map $Y \xrightarrow{\chi_m} \Omega$ such that

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ Y \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback square

(Equivalently, a partial map classifier for 1 !)

It follows (with effort!) that every elementary topos $\mathcal{E}$

- has finite colimits
- is <u>locally</u> <u>cartesian closed</u>:

  every slice category $\mathcal{E}/X$ is cartesian closed

- all monos and epis are regular
- every $X \xrightarrow{f} Y$ factors as $f = X \overset{e}{\twoheadrightarrow} Z \overset{m}{\rightarrowtail} Y$ where e is epi and m mono
- in every pullback square with e epi

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "e"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

  - e' is also epi
  - f' mono $\Rightarrow$ f mono

A category $\mathcal{E}$ is a Grothendieck topos if:

- it is an elementary topos
- it has coproducts ($\equiv$ ly is cocomplete)
- it is locally small
- and it has a <u>set of generators</u>:

there exists a set $\mathcal{G} \subseteq |\mathcal{E}|$

such that for any parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{E}$

we have

$$f = g \iff \forall A \in \mathcal{G} \text{ and } A \xrightarrow{x} X \quad f \circ x = g \circ x$$

Set is a Grothendieck topos

The subobject classifier is $1 \xrightarrow{* \mapsto \text{true}} \{\text{true}, \text{false}\}$

Given $X \overset{m}{\rightarrowtail} Y$, the unique $Y \xrightarrow{\chi_m} \{\text{true}, \text{false}\}$

s.t.
$$
\begin{tikzcd}
X \arrow[r, "!"] \arrow[d, rightarrowtail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "* \mapsto \text{true}"] \\
Y \arrow[r, "\chi_m"'] & \{\text{true}, \text{false}\}
\end{tikzcd}
$$
is $\chi_m(y) = \begin{cases} \text{true} & \text{if } y \in m(X) \\ \text{false} & \text{if } y \notin m(X) \end{cases}$

$\{1\}$ is a (singleton) set of generators

(1 is a generating object.)

$Psh(\mathbb{C})$ is a Grothendieck topos, for any small category $\mathbb{C}$.

A <u>sieve</u> on $X \in |\mathbb{C}|$ is a set S of maps in C with codomain X s.t.

$$Y \xrightarrow{f} X \in S \text{ and } Z \xrightarrow{g} Y \text{ in } \mathbb{C} \Rightarrow Z \xrightarrow{g \circ f} X \in S$$

$\Omega(X) :=$ the set of all sieves on X

$$\begin{tikzcd} X & & \Omega(X) \arrow[d] \\ Y \arrow[u, "f"] & \longmapsto & \Omega(Y) \end{tikzcd} \quad S \mapsto \{Z \xrightarrow{g} Y \mid Z \xrightarrow{f \circ g} X \in S\}$$

Given a mono $P \overset{m}{\rightarrowtail} Q$ in $Psh(\mathbb{C})$

$Q \xrightarrow{\chi_m} \Omega$ defined by:

notation for $Q(f)(y)$

$$(\chi_m)_X := y \mapsto \{Z \xrightarrow{f} X \mid \exists x \in P(Z).\ m_Z(x) = y \cdot f\} : Q(X) \to \Omega(X)$$

is unique such that

$$\begin{tikzcd} P \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ Q \arrow[r, "\chi_m"'] & \Omega \end{tikzcd} \qquad \top_X := * \mapsto \{Z \xrightarrow{f} X \mid true\}$$

the maximum sieve

A set of generators is

$$G := \{ \underline{y}(X) \mid X \in |\mathbb{C}| \}$$

the set of all representables

Proof that G is a generating set.

Suppose $P \overset{p}{\underset{q}{\rightrightarrows}} Q$ is a parallel pair in $Psh(\mathbb{C})$

such that

$$\forall X \in |\mathbb{C}|, \forall \underline{y}X \xrightarrow{x} P \text{ in } Psh(\mathbb{C}), \quad p \circ x = q \circ x$$

By Yoneda, this says

$$\forall X \in |\mathbb{C}| \quad \forall x \in P(X) \quad p_X(x) = q_X(x).$$

I.e. $p = q$.

$Sh(S)$ is a Grothendieck topos, for any top. space S.

A key preliminary result is

<u>Theorem</u> The inclusion functor

$$Sh(S) \xhookrightarrow{I} PSh(\mathcal{O}(S))$$

has a left-exact left adjoint

preserves finite limits

$$PSh(\mathcal{O}(S)) \xrightarrow{a} Sh(S)$$

the <u>associated sheaf</u> (or <u>sheafification</u>) functor.

Grothendieck's $(\cdot)^+$ Functor $Psh(\mathcal{O}(S)) \to Psh(\mathcal{O}(S))$

Given a presheaf $P : \mathcal{O}(S)^{op} \to \underline{Set}$

Define $P^+(U)$ = equivalence classes of matching families covering U

Recall a matching family covering U is

$$\left(x_i \in P(U_i)\right)_{i \in I} \quad \text{for some } (U_i)_{i \in I} \text{ with } \bigcup_{i \in I} U_i = U$$

such that, for all $i, j \in I$, $x_i\restriction_{U_i \cap U_j} = x_j\restriction_{U_i \cap U_j}$

Define

$$\left(x_i \in P(U_i)\right)_{i \in I} \sim \left(x_j' \in P(U_j')\right)_{j \in J}$$

to hold if $\forall i \in I, j \in J$ $\quad x_i\restriction_{U_i \cap U_j'} = x_j'\restriction_{U_i \cap U_j'}$

Exercise: • Work out the morphism action of P^+ and check that it respects the equivalence relation.

• Work out the morphism action of the functor $(\cdot)^+$

The functions

$$x \in P(U) \longmapsto [(x \in P(U))]_\sim$$

indexed by $U \in \mathcal{O}(X)$ define a natural transformation $P \to P^+$

We say P is <u>separated</u> if every matching family has at most one amalgamation.

(Recall a presheaf P is a <u>sheaf</u> if every matching family has a unique amalgamation.)

<u>Lemma</u> (Grothendieck)

1) The functor $(\cdot)^+ : \mathrm{Psh}(\mathcal{O}(X)) \to \mathrm{Psh}(\mathcal{O}(X))$ preserves finite limits.

2) Given a sheaf Q and presheaf map $P \xrightarrow{f} Q$, there is a unique presheaf map $P^+ \xrightarrow{\bar{f}} Q$ s.t.

$$\begin{tikzcd} P^+ \arrow[r, "\bar{f}"] & Q \\ P \arrow[u, "\xi"] \arrow[ur, "f"'] & \end{tikzcd}$$

3) For every presheaf P, P^+ is separated.

4) If P is separated then P^+ is a sheaf.

It follows immediately from the lemma that $\underline{a} := (\ )^{++}$ defines a left-exact left adjoint to.

$I : Sh(S) \hookrightarrow Psh(O(S))$

<u>A subtle point</u>. I can be written as a composition of inclusion functors with left adjoints (in black)

separated presheaves

$$\begin{tikzcd}[column sep=large]
Sh(S) \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=4, "\top"] & Sep(S) \arrow[l, bend right, "(\cdot)^+"'] \arrow[r, hook] \arrow[r, phantom, shift left=3, "\perp"] & Psh(O(S)) \arrow[l, bend right, "\underline{b}"'] \arrow[l, bend right=70, "(\cdot)^+"'] \arrow[ll, bend left=40, "\underline{a}"]
\end{tikzcd}$$

We also have a functor $(\cdot)^+ : Psh(O(S)) \to Sep(S)$

Since adjoints compose $\underline{a} \cong (\cdot)^+ \circ \underline{b}$

and by definition $\underline{a} = (\cdot)^+ \circ (\cdot)^+$

However, $\underline{b} \not\cong (\cdot)^+ : Psh(O(S)) \to Sep(S)$

An advantage of $(\cdot)^+$ is it preserves finite limits ($\underline{b}$ doesn't!)

We are looking at a special type of adjunction

A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be <u>reflective</u> if the inclusion functor $\mathcal{C}' \overset{I}{\hookrightarrow} \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{C}'$.

In such an adjunction, the counit $\varepsilon : FI \Rightarrow 1_{\mathcal{C}'}$ is a natural isomorphism and, for any $Y \in |\mathcal{C}'|$, the component $Y \xrightarrow{\eta_Y} IFY$ of the unit $\eta : 1_{\mathcal{C}} \Rightarrow IF$ is an iso.

These observations are generalised by:

<u>Proposition</u> Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with counit $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ and unit $\eta : 1_{\mathcal{C}} \Rightarrow GF$.

1) G is faithful iff every component of ε is an epi in $\mathcal{D}$.
2) G is full iff every component of ε is a split mono in $\mathcal{D}$.
3) G is full & faithful iff every ε component is an iso in $\mathcal{D}$.

If G is full and faithful then define $\mathcal{C}'$ to be the full subcategory of $\mathcal{C}$ on objects X such that η_X is an iso. Then $FI : \mathcal{C}' \to \mathcal{D}$ is an equivalence of categories and

$$\begin{tikzcd} \mathcal{C}' \arrow[r, hook, "I"] \arrow[d, "FI"'] \arrow[dr, phantom, "\overset{\cong}{\Swarrow} \eta I"] & \mathcal{C} \\ \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

The full subcategory $\mathcal{C}'$ in the proposition enjoys the property that it is <u>replete</u>:

$$X \in |\mathcal{C}'|,\ Y \in |\mathcal{C}|,\ X \cong Y \Rightarrow Y \in \mathcal{C}'.$$

By the proposition, any adjunction with full and faithful right adjoint (in particular any full reflective subcategory) is equivalent to a replete full reflective subcategory.

Naturally occurring examples of full reflective subcategories are often replete by definition

e.g.

$$Sh(S) \hookrightarrow Psh(\mathcal{O}(S))$$

$$Sep(S) \hookrightarrow Psh(\mathcal{O}(S))$$

$$Sh(S) \hookrightarrow Sep(S).$$

Suppose S is a reflective replete full subcategory of $\mathcal{C}$ with reflection $F \dashv I : S \hookrightarrow \mathcal{C}$.

Proposition I creates limits.

It follows that any reflective full subcategory of a complete category is complete, with limits calculated as in the super-category.

Proposition If a diagram $D : G \to S$ has colimit $\varinjlim D$ in $\mathcal{C}$ then it has colimit $F(\varinjlim D) \circ (\eta I D)$ in S.

So any full reflective subcategory of a cocomplete category is cocomplete, with colimits calculated by reflecting colimits in the supercategory.

(Continuing with the assumptions of the previous page.)

<u>Proposition</u> If $\mathcal{C}$ is cartesian closed and $F : \mathcal{C} \to \mathcal{S}$ preserves finite products then $\mathcal{S}$ is an <u>exponential ideal</u> of $\mathcal{C}$:

$$Y \in |\mathcal{S}|, \; X \in |\mathcal{C}| \Rightarrow [X, Y] \in \mathcal{S}$$

In particular $\mathcal{S}$ is cartesian closed and the inclusion $\mathcal{S} \xhookrightarrow{I} \mathcal{C}$ preserves cartesian closed structure.

Returning to

$$\begin{tikzcd} Sh(S) \arrow[r, hook, "I"'] \arrow[r, phantom, "\perp"] & Psh(\mathcal{O}(S)) \arrow[l, bend right, "a"'] \end{tikzcd}$$

By Lecture 11, $Psh(\mathcal{O}(S))$ is complete, cocomplete and cartesian closed.

By the above, $Sh(S)$ is also complete, cocomplete and cartesian closed.

The subobject classifier in $Sh(S)$

$$\Omega_S(U) := \{U' \in \mathcal{O}(S) \mid U' \subseteq U\}$$

$$\begin{array}{ccc} U & & \Omega_S(U) \\ \cup\!\shortmid & \longmapsto & \downarrow \; (U' \mapsto U' \cap V) \\ V & & \Omega_S(V) \end{array}$$

This is a sheaf. It is isomorphic to the sheaf C_{Si} of continuous functions into Sierpinski space $S_i = \{\bot, \top\}$ $\mathcal{O}(S_i) = \{\emptyset, \{\top\}, \{\bot, \top\}\}$

$1 \xrightarrow{\top} \Omega$

$$\top_U := * \mapsto U : 1_U \to \Omega_S(U)$$

Exercise

Verify that the above indeed defines a subobject classifier in $Sh(S)$.

Key Point Consider any subpresheaf P

$$P \subseteq Q$$

of a sheaf Q,

the components of the mono are all subset inclusions

characterise when it holds that P is also a sheaf.

$$\boxed{\text{Sh(S) is indeed an elementary topos}}$$

Proposition Every representable $\underline{y}(U)$ is a sheaf.

Since the set of representables is a generating set in $Psh(\mathcal{O}(S))$, it is also a generating set in $Sh(S)$.

$$\boxed{Sh(S) \text{ is a Grothendieck topos!}}$$

Exercise
- Every representable is isomorphic to a unique subsheaf of 1.
- Every subsheaf of 1 is isomorphic to a unique representable.

Representables coincide with subterminal objects, and $Sh(S)$ has a generating set of subterminals.

Other example families of toposes

- Sheaves on a <u>locale</u>. Grothendieck toposes generalising sheaves on a topological space.
- Sheaves on a <u>site</u> (small category + Grothendieck topology). These are exactly the Grothendieck toposes.
- <u>Realisability toposes</u>. Elementary (but not Grothendieck) toposes related to logic and computability theory.

Higher-dimensional analogues of Grothendieck toposes are important in topology; e.g.,

- <u>Infinity toposes</u> of Jacob Lurie.